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Definition (Lie Group)

A *Lie group* G is a smooth (finite dimensional) manifold equipped with a smooth operation $* : G \times G \to G$ such that $(G, *)$ is group.

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Prime examples.

- $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$ and $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$
- $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$
- $\mathrm{O}_n(\mathbb{R})$, $SO_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$
- U_n and SU_n
- $\mathrm{UPT}_n(\mathbb{R})$ and $\mathrm{UPT}_n(\mathbb{C})$

Recall. Every open subset of $\mathbb{R}^n$, $\mathbb{C}^n$ or, more general, of any finite dimensional real/complex vector space is a smooth manifold.

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Two more advanced examples.

- Let M be a compact Riemannian manifold and let $\mathrm{Diff}(M)$ be the set of all smooth diffeomorphisms of M. Then $\mathrm{Diff}(M)$ has a natural infinite dimensional (Fréchet) Lie group structure (cf. Neeb, *Towards a Lie theory of locally convex groups*, 2006).
- Moreover, let $\mathrm{I}(M) \subset \mathrm{Diff}(M)$ be the subgroup of all isometry of M. Then $\mathrm{I}(M)$ carries the structure of a finite dimensional Lie group (Myers & Steenrod, *The group of isometries of a Riemannian manifold*, 1939).

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Simplification. In this lecture (part I) we want to restrict our considerations to so-called *linear Lie groups*, i.e. to *Lie subgroups* of $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$ or, more general, $\mathrm{GL}(V)$, where V is a finite dimensional real/complex vector space.

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Answer.

- It is definitely a restriction in the sense that not all finite dimensional Lie groups can be represented as *Lie subgroups* of some $\mathrm{GL}(V)$ (counter-example: Double cover of $\mathrm{SL}_2(\mathbb{R})$, cf. Knapp, Chap. VI, Sec. 3)
- However, on the "infinitesimal level" it does not yield any restriction due to Ado's theorem.

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- However, on the "infinitesimal level" it does not yield any restriction due to Ado's theorem.

Theorem (Ado, 1935)

Every finite dimensional real/complex Lie algebra can be represented as a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional real/complex vector and hence as a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{C})$ for some $n \in \mathbb{N}$.

Definition (Lie subgroup)

A Lie subgroup of $\mathrm{GL}(V)$ is a subgroup of $\mathrm{GL}(V)$ which carries the structure of an immersed submanifold of $\mathrm{GL}(V)$.

Examples.

- $\mathrm{SL}_n(\mathbb{R})$, $O_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, $\mathrm{UPT}_n(\mathbb{R})$ are Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$
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- Let α be irrational and consider the following subgroup of $\mathrm{GL}_4(\mathbb{R})$:

$$W_\alpha := \left\{ \begin{pmatrix} \cos t & \sin t & 0 & 0 \\ -\sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos(\alpha t) & \sin(\alpha t) \\ 0 & 0 & -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix} \middle| \, t \in \mathbb{R} \right\} \subset \left\{ \begin{pmatrix} \Theta & 0_2 \\ 0_2 & \Omega \end{pmatrix} \middle| \, \Theta, \Omega \in SO(2) \right\} =: T^2 \subset \mathrm{GL}_4(\mathbb{R}) .$$

W_α is a 1-dimensional immersed submanifold and thus a Lie subgroup of $\mathrm{GL}_4(\mathbb{R})$.

Definition (Lie algebra)

A (finite dimensional) real/complex Lie algebra $\mathfrak{g}$ is a real/complex (finite dimensional) vector space equipped with a bilinear, skew-symmetric operation $(A, B) \mapsto [A, B]$ which satisfies the Jacobi-identity, i.e.

- $[\lambda A + C, B] = \lambda[A, B] + [C, B] \quad$ for all $\quad A, B, C \in \mathfrak{g}. \quad \lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$)
- $[A, B] = -[B, A] \quad$ for all $\quad A, B \in \mathfrak{g}$.
- $\big[A, [B, C]\big] = \big[[A, B], C\big] + \big[B, [A, C]\big] \quad$ for all $\quad A, B, C \in \mathfrak{g}$.

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Examples.

- $\mathfrak{gl}_n(\mathbb{R}) = \mathbb{R}^{n\times n}$, $\mathfrak{gl}_n(\mathbb{C}) = \mathbb{R}^{n\times n}$
- $\mathfrak{sl}_n(\mathbb{R}) = \ldots$, $\mathfrak{sl}_n(\mathbb{C}) = \ldots$
- $\mathfrak{so}_n(\mathbb{R}) = \ldots$, $\mathfrak{u}_n = ..$, $\mathfrak{su}_n = ..$

} equipped with the commutator $[A, B] := AB - BA$.

- $\mathbb{R}^n$ or $\mathbb{C}^n$ equipped with $[A, B] = 0$ (Abelian Lie algebra).

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- $[\lambda A + C, B] = \lambda[A, B] + [C, B]$ for all $A, B, C \in \mathfrak{g}$. $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$)
- $[A, B] = -[B, A]$ for all $A, B \in \mathfrak{g}$.
- $\big[A, [B, C]\big] = \big[[A, B], C\big] + \big[B, [A, C]\big]$ for all $A, B, C \in \mathfrak{g}$.

In what follows we focus on real Lie algebras!

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Lie's Theorem

Let G be a finite dimensional Lie group.

(a) The tangent space $\mathrm{T}_{\mathrm{I}}G$ of G at the identity I carries a natural Lie algebra structure.
(b) The tangent space $\mathrm{T}_{\mathrm{I}}H$ of every Lie subgroup of $H \subset G$ is a Lie subalgebra of $\mathrm{T}_{\mathrm{I}}G$.
(c) Conversely, to every Lie subalgebra $\mathfrak{h}$ of $\mathrm{T}_{\mathrm{I}}G$ there is a unique path-connected Lie subgroup with Lie algebra $\mathfrak{h}$.

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(c) Conversely, to every Lie subalgebra $\mathfrak{h}$ of $\mathrm{T_I}G$ there is a unique path-connected Lie subgroup with Lie algebra $\mathfrak{h}$.

Proof sketch:

(a) & (b) Use the following facts (see also "Running Example"): $\mathrm{T_I}H \subset \mathrm{T_I}G$ and

$$\begin{aligned} \mathrm{T_I}G &\cong \text{set of all right-invariant vector fields on } G \\ &= \text{Lie subalgebra of the Lie algebra of all smooth vector fields on } G \end{aligned}$$

(c) Apply the Frobenius' theorem to the (right invariant) "distribution" generated by $\mathfrak{h}$.

Running Example: $\mathrm{SO}_n(\mathbb{R})$

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- Subgroup of $\mathrm{GL}_n(\mathbb{R})$: ✓
- Submanifold: Consider the map

$$\Phi : \mathbb{R}^{n\times n} \to \mathrm{Sym}_n(\mathbb{R})\,, \quad \Phi(X) := X^\top X - I_n\,.$$

Obviously, $O_n(\mathbb{R}) = \Phi^{-1}(0_n)$. Thus, the regular value theorem (RVT) – if applicable – would imply that $O_n(\mathbb{R})$ is a submanifold of $\mathrm{GL}_n(\mathbb{R}) \subset \mathbb{R}^{n\times n}$. Therefore, let's investigate whether $0_n \in \mathrm{Sym}_n(\mathbb{R})$ is a regular value of Φ:

- Derivative of Φ: $\mathrm{D}\Phi(X)H = H^\top X + X^\top H, \quad X, H \in \mathbb{R}^{n\times n}$
- Let $X \in \Phi^{-1}(0_n)$ and $S \in \mathrm{Sym}_n(\mathbb{R})$. Choose $H := \frac{1}{2}XS$ then

$$\mathrm{D}\Phi(X)H = \frac{1}{2}(XS)^\top X + \frac{1}{2}X^\top(XS) = \frac{1}{2}SX^\top X + \frac{1}{2}X^\top XS = S\,,$$

i.e. $\mathbb{D}\Phi(X)$ is surjective for all $X \in \Phi^{-1}(0_n)$ and thus RVT applies.
- Hence $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R}) = \mathrm{O}_n(\mathbb{R}) \cap \mathrm{GL}_n^+(\mathbb{R})$ are submanifolds of $\mathrm{GL}_n(\mathbb{R}) \subset \mathbb{R}^{n\times n}$.
- Moreover, $O_n(\mathbb{R})$ and $SO_N(\mathbb{R})$ are closed and bounded and thus compact.

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- Subgroup of $\mathrm{GL}_n(\mathbb{R})$: ✓
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- Tangent space at the identity I_n: By RVT we know

$$\mathrm{T}_{\mathrm{I}_n}\mathrm{O}_n(\mathbb{R}) = \ker \mathrm{D}\Phi(\mathrm{I}_n) = \{H \in \mathbb{R}^{n\times n} \mid H^\top + H = 0\} = \mathfrak{so}_n(\mathbb{R})\,.$$

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- Tangent space at the identity I_n: ✓
- Lie algebra structure: Obviously, $\mathfrak{so}_n(\mathbb{R})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$. But can we argue a bit more general as in Lie's theorem?

Running Example: $\mathrm{SO}_n(\mathbb{R})$

- Subgroup of $\mathrm{GL}_n(\mathbb{R})$: ✓
- Submanifold: ✓
- Tangent space at the identity I_n: ✓
- Lie algebra structure: Consider the right "translation" r_Θ on $\mathrm{SO}_n(\mathbb{R})$ by $\Theta \in \mathrm{SO}_n(\mathbb{R})$, i.e.

$$r_\Theta(X) := X\Theta\,.$$

Obviously, $r_\Theta : \mathrm{SO}_n(\mathbb{R}) \to \mathrm{SO}_n(\mathbb{R})$ yields a diffeomorphism which maps I to Θ. Hence

$$\mathrm{D}r_\Theta(\mathrm{I}_n) : \mathrm{T}_{\mathrm{I}_n}\mathrm{SO}_n(\mathbb{R}) \to \mathrm{T}_\Theta\mathrm{SO}_n(\mathbb{R})\,.$$

This allows us to define so-called right-invariant vector fields on $\mathrm{SO}_n(\mathbb{R})$ via

$$\xi_A(\Theta) := \mathrm{D}r_\Theta(\mathrm{I})A \quad \text{with} \quad A \in T_{\mathrm{I}_n}\mathrm{SO}_n(\mathbb{R})\,. \tag{$*$}$$

To make $(*)$ more explicit we compute the derivative $\mathrm{D}r_\Theta(\mathrm{I}_n)A = A\Theta$. Hence we obtain

$$\xi_A(\Theta) := A\Theta\,.$$

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Finally, let us compute the Lie derivative of two right-invariant vector fields:

$$\mathrm{L}_{\xi_B}\xi_A(\Theta) := \mathrm{D}\xi_A(\Theta)\xi_B(\Theta) - \mathrm{D}\xi_B(\Theta)\xi_A(\Theta) = AB\Theta - BA\Theta = [AB - BA]\Theta$$

Since it is well known that $\mathrm{L}_{\xi_B}\xi_A$ is again a vector field on the respective submanifold (here $\mathrm{SO}_n(\mathbb{R})$) we conclude that $[AB - BA] \in \mathrm{T}_{\mathrm{I}_n}\mathrm{SO}_n(\mathbb{R})$. i.e. $\mathrm{T}_{\mathrm{I}_n}\mathrm{SO}_n(\mathbb{R})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{R})$.

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For now end of running example!

Concluding remarks and useful results.

Theorem

Let G be an arbitrary Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ with Lie subalgebra $\mathfrak{g} := \mathrm{T}_\mathrm{I} G$. Then one has the following identities

(a) *Trotter formula:* $\lim_{n\to\infty} \left(\mathrm{e}^{A/n}\mathrm{e}^{B/n}\right)^n = \mathrm{e}^{A+B}$ *for all* $A, B \in \mathfrak{g}$.

(b) *Commutator formula:* $\lim_{n\to\infty} \left(\mathrm{e}^{A/n}\mathrm{e}^{B/n}\mathrm{e}^{-A/n}\mathrm{e}^{-B/n}\right)^{n^2} = [A, B]$ *for all* $A, B \in \mathfrak{g}$.

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Yamabe's Theorem

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References:

- Hilgert/Neeb, *Structure and Geometry of Lie Groups*, Springer, 2012.
- Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, 2nd ed., 2002.

The following type of control system is usually called a **bilinear system**

$$\dot{x}(t) = Ax(t) + \underbrace{\sum_{k=1}^{m} u_k(t)B_k x(t)}_{\text{bilinear term}}, \quad x(0) = x_0 \in \mathbb{R}^n \tag{Σ}$$

with $A, B \in \mathbb{R}^{n\times n}$ and $x(t) \in \mathbb{R}^n$ and $u_k(\cdot) \in PC\big(\mathbb{R}, \mathbb{R}\big)$ for $k =, 1. \dots, m$.

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Define the **group lift** of (Σ) as follows

$$\dot{X}(t) = \Big(A + \sum_{k=1}^{m} u_k(t) B_k\Big) X(t), \quad X(0) = X_0 \in \mathbb{R}^{n\times n} \tag{$\widehat{\Sigma}$}$$

with $A, B \in \mathbb{R}^{n\times n}$ and $X(t) \in \mathbb{R}^{n\times n}$ and $u_k(\cdot) \in PC\big(\mathbb{R},\mathbb{R}\big)$ for $k =, 1. \dots, m$.

Reachable sets of (Σ) and $(\widehat{\Sigma})$:

Denote by $t \mapsto x(t, x_0, u)$ and $t \mapsto X(t, X_0, u)$ the unique solutions of (Σ) and $(\widehat{\Sigma})$ corresponding to the control $u(\cdot)$ and the initial value x_0 and X_0, respectively.

Note: Uniqueness follows immediately from Picard-Lindelöf and the linear boundedness of the right side implies that $t \mapsto x(t, x_0, u)$ and $t \mapsto X(t, X_0, u)$ exist for all $t \geq 0$.

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We define the following different types of reachable set:

$$
\begin{aligned}
R_\Sigma(x_0, T) &:= \{x(T, x_0, u) \mid u \in PC\big(\mathbb{R}, \mathbb{R}^m\big)\}, \\
R_{\widehat{\Sigma}}(X_0, T) &:= \{X(T, X_0, u) \mid u \in PC\big(\mathbb{R}, \mathbb{R}^m\big)\}
\end{aligned}
$$

and

$$
R_\Sigma(x_0, \leq T) := \bigcup_{0 \leq t \leq T} R_\Sigma(x_0, t), \quad R_\Sigma(x_0) := \bigcup_{0 \leq t < \infty} R_\Sigma(x_0, t)
$$

and $R_{\widehat{\Sigma}}(X_0, \leq T)$, $R_{\widehat{\Sigma}}(X_0)$, respectively.

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- Given a fixed $u(\cdot)$ and the corresponding solution $X(t, \mathrm{I}_n, u)$ of $(\widehat{\Sigma})$ then $X(t, \mathrm{I}_n u)x_0$ yields the solution of (Σ) with initial value x_0;
- Hence the reachable sets $R_\Sigma(x_0, T)$ $R_\Sigma(x_0, \leq T)$ and $R_\Sigma(x_0)$ of (Σ) are related to the corresponding reachable sets of $(\widehat{\Sigma})$ as follows:

$$R_\Sigma(x_0, T) = R_{\widehat{\Sigma}}(\mathrm{I}_n, T)x_0\,, \quad R_\Sigma(x_0, \leq T) = R_{\widehat{\Sigma}}(\mathrm{I}_n, \leq T)x_0\,, \quad R_\Sigma(x_0) = R_{\widehat{\Sigma}}(\mathrm{I}_n)x_0\,.$$

Bilinear Control Systems – Basic Properties

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- The reachable set $R_{\widehat{\Sigma}}(\mathrm{I}_n) \subset \mathrm{GL}_n(\mathbb{R})$ takes the form of a (Lie) semigroup, i.e.

$$\mathrm{I}_n \in R_{\widehat{\Sigma}}(\mathrm{I}_n) \quad \text{and} \quad X_1, X_2 \in R_{\widehat{\Sigma}}(\mathrm{I}_n) \quad \Longrightarrow \quad X_1 X_2 \in R_{\widehat{\Sigma}}(\mathrm{I}_n)\,.$$

The last property suggests to apply Lie group theory for analyzing reachable sets of bilinear systems.

Bilinear Control Systems – Basic Properties

Recall: What is an (right) invariant system on a Lie group G?

Let $\mathfrak{g}$ be the Lie algebra of G and let $r_g : G \to G$ denote right-multiplication by g.

- A vector field ξ on G is called right-invariant if it takes the form

$$\xi(g) = \mathrm{D}r_g(e)\,A\,, \quad \text{where } A \text{ is a fixed element of } \mathfrak{g}\,.$$

- A control system is called right-invariant if it takes the form

$$\dot{g} = \xi(g,u)\,, \quad \text{such that all vector fields } \xi(g,u) \text{ are right-invariant}\,.$$

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Bottom line: $(\widehat{\Sigma})$ constitutes a right-invariant systems on $G = \mathrm{GL}_n(\mathbb{R})$

In what follows G will always denote a Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{g}$ its Lie subalgebra in $\mathfrak{gl}_n(\mathbb{R})$ or $\mathfrak{gl}_n(\mathbb{C})$.

Bilinear Control Systems – Basic Properties

For analyzing the **fundamental properties** of bilinear systems we need on more concept:

The *orbit* $\mathcal{O}_{\widehat{\Sigma}}(\mathrm{I}_n)$ of I_n with respect to $(\widehat{\Sigma})$ is given by

$$\mathcal{O}_{\widehat{\Sigma}}(\mathrm{I}_n) := \bigcup_{t \in \mathbb{R}} \{X(t, \mathrm{I}_n, u) \mid u \in PC\big(\mathbb{R}, \mathbb{R}^m\big)\}\,.$$

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Fundamental Theorem I (Jurdjevic & Sussmann 1972)

The orbit of the lifted system

$$\dot{X}(t) = \Big(A + \sum_{k=1}^{m} u_k(t) B_k\Big) X(t) \,. \qquad (\widehat{\Sigma})$$

is a Lie subgroup with Lie algebra $\langle A, B_1, \ldots, B_m\rangle_{\mathrm{L}}$, where $\langle A, B_1, \ldots, B_m\rangle_{\mathrm{L}}$ denotes the real Lie algebra generated by $A, B_1, \ldots, B_m$ (via iterated commutators).

Bilinear Control Systems – Basic Properties

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Proof sketch:

- Obviously, the orbit $\mathcal{O}_{\widehat{\Sigma}}(\mathrm{I}_n)$ is a path-connected subgroup of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ and hence by Yamabe's theorem a Lie subgroup. Let $\mathfrak{g}$ be denote the Lie subalgebra of $\mathcal{O}_{\widehat{\Sigma}}(\mathrm{I}_n)$ and let $G(A, B_1, ..., B_m)$ be the unique path-connected Lie subgroup corresponding to $\langle A, B_1, ..., B_m \rangle_{\mathrm{L}}$.
- Then $\mathcal{O}_{\widehat{\Sigma}}(\mathrm{I}_n) \subset G(A, B_1, \dots, B_m)$ implies $\mathfrak{g} \subset \langle A, B_1, \dots, B_m \rangle_{\mathrm{L}}$.
- Conversely, e^{At} and $\mathrm{e}^{(A+B_k)t}$, $k = 1, \dots, m$ are in $\mathcal{O}_{\widehat{\Sigma}}(\mathrm{I}_n)$ and thus $A, A + B_1, \dots, A + B_m \in \mathfrak{g}$. Since $\mathfrak{g}$ is a Lie subalgebra we conclude $A, B_1, \dots, B_m \in \mathfrak{g}$ and $\langle A, B_1, \dots, B_m \rangle_{\mathrm{L}} \subset \mathfrak{g}$.

Definition: Let G be a path-connected Lie subgroup $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ with Lie algebra $\mathfrak{g}$ and let $(\widehat{\Sigma})$ be the above bilinear system with $A, B_1, \dots, B_m \in \mathfrak{g}$.

- $(\widehat{\Sigma})$ is called controllable (w.r.t. to G) if $R_{\widehat{\Sigma}}(\mathrm{I}_n) = G$.
- $(\widehat{\Sigma})$ is called accessible (w.r.t. to G) if $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ has interior points (w.r.t. to G).
- $(\widehat{\Sigma})$ is called strongly accessible if $R_{\widehat{\Sigma}}(\mathrm{I}_n), \leq T$ has interior points for all $T > 0$.

Bilinear Control Systems – Basic Properties

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Fundamental Theorem II (Jurdjevic & Sussmann 1972; Brockett 1973)

Given a bilinear systems $(\widehat{\Sigma})$ on G with $A, B_1, \dots, B_m \in \mathfrak{g}$. Then one has

(a) $\langle A, B_1, \dots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g} \iff (\widehat{\Sigma})$ is strongly accessible. $\iff (\widehat{\Sigma})$ is accessible.

(c) $\langle A, B_1, \dots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g}$ and $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ is a group. $\iff (\widehat{\Sigma})$ is controllable.

(c) $\langle B_1, \dots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g} \implies (\widehat{\Sigma})$ is controllable.

(d) If $G \subset \mathrm{GL}_n(\mathbb{R})$ is additionally compact, the one has the equivalence:

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Proof sketch. (a) $\Longrightarrow$: Assume $\langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g}$ and consider maps of the form

$$(t_1, \ldots, t_r) \mapsto \mathrm{e}^{t_1 C_1} \cdots \mathrm{e}^{t_r C_r}$$

with C_l of the from $A + \sum_{k=1}^{m} u_k B_k$, $t_l > 0$ and $\sum_{l=1}^{r} t_l \leq T$. Then one can prove that there is such a map with rank equal to $\dim \mathfrak{g}$. This implies strong accessibility

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Proof sketch. (a) $\implies$: ✓

$\implies$: Assume $(\widehat{\Sigma})$ accessibility then the orbit $\mathcal{O}_{\widehat{\Sigma}}(\mathrm{I}_n)$ is equal to G, cf. proof of part (b). Then we conclude by FT I $\langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g}$.

Bilinear Control Systems – Basic Properties

Fundamental Theorem II (Jurdjevic & Sussmann 1972; Brockett 1973)

Given a bilinear systems $(\widehat{\Sigma})$ on G with $A, B_1, \ldots, B_m \in \mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$. Then one has

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$$\langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g} \iff (\widehat{\Sigma}) \text{ is controllable.}$$

Proof sketch. (b) $\Longleftarrow$: ✓ (use part (a))

$\Longrightarrow$: Assume $\langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g}$ and that $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ is a group. Then by part (a) $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ contains interior point. Let X_* be such an interior point in $R_{\widehat{\Sigma}}(\mathrm{I}_n)$. Then $\mathcal{U} := R_{\widehat{\Sigma}}(\mathrm{I}_n)X_*^{-1}$ is equal to $R_{\widehat{\Sigma}}(\mathrm{I}_n)X_*^{-1}$ and moreover a neighborhood of I_n. Thus a standard result implies that $\bigcup_{k \in \mathbb{N}} \mathcal{U}^n$ is equal to G and hence $G = R_{\widehat{\Sigma}}(\mathrm{I}_n)$.

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(d) If $G \subset \mathrm{GL}_n(\mathbb{R})$ is additionally compact, the one has the equivalence:

$$\langle A, B_1, \dots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g} \iff (\widehat{\Sigma}) \text{ is controllable.}$$

Proof sketch. (c) $\Longrightarrow$: First, let $\widetilde{\Sigma}$ denote the system which results by setting $A = 0$. Then the orbit $\mathcal{O}_{\widetilde{\Sigma}}(\mathrm{I}_n)$ is a group and thus by part (b) equal to G. Moreover, one has

$$\lim_{\tau \to 0} \mathrm{e}^{\tau(A \pm \tau^{-1} t B_k)} = \mathrm{e}^{\pm t B_k}$$

and hence the closure of $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ (w.r. to G) is equal to G. Thus $R_{\widehat{\Sigma}}(\mathrm{I}_n) = G$, cf. proof of part (d).

Bilinear Control Systems – Basic Properties

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Proof sketch. (d) $\Longleftarrow$: ✓

$\Longrightarrow$: Assume $\langle A, B_1, \ldots, B_m \rangle_{\mathrm{LA}} = \mathfrak{g}$ and that G is compact. Consider the closure of $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ a standard result says that any closed semigroup of a compact Lie group is already a group. Hence the closure of $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ is equal to G. Finally, we have to show that this equality holds even without closure.

Bilinear Control Systems – Basic Properties

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Proof sketch. (d) $\Longleftarrow$: ✓

$\implies$: To this end choose $X_0, X_1 \in G$ and let the system evolve "forward" in time from X_0 and "backward" in time from X_1. By accessibility both reachable set contain interior point which can be steered from one to the other (due to the fact the $R_{\widehat{\Sigma}}(\mathrm{I}_n)$ is dense in G). Concatenating the first to controls with the "reverse" of the third one yields the desired result.

Bilinear Control Systems – Basic Properties

Fundamental Theorem II (Jurdjevic & Sussmann 1972; Brockett 1973)

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Remark. Note the for accessibility and controllability aspects piecewise constant controls are perfectly fine in the sense that one cannot improve accessibility and controllability by passing to a larger class of controls.

However, for optimal control issues piecewise constant controls are i.g. not adequate.

Running example cont. Consider the following bilinear system:

$$\dot{X}(t) = (A + u(t)B)X(t) \tag{1}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

- A, B are skew-symmetric and thus $A, B \in \mathfrak{so}_3(\mathbb{R})$.
- Hence we can consider (1) as a system on $\mathrm{SO}_3(\mathbb{R})$.
- Clearly, $A, B, [A, B]$ is a basis of $\mathfrak{so}_3(\mathbb{R})$.
- Thus $\langle A, B\rangle_{\mathrm{LA}} = \mathfrak{so}_3(\mathbb{R})$ and thus (1) is accessible.
- $SO_3(\mathbb{R}) \subset \mathrm{GL}_3(\mathbb{R})$ is compact and thus (1) is even controllable.

How does FT II help us to determine controllability of the “original” bilinear systems (Σ)?

Let us investigate this a bit more general.

Group action. Let G be an arbitrary Lie group and M a smooth manifold. A *group action* of G on M is a map $\alpha : G \times M \to M$ such that the following holds:

$$\alpha(\mathrm{I}, p) = p \quad \text{and} \quad \alpha\big(g, \alpha(h, p)\big) = \alpha(gh, p) \quad \text{for all } g, h \in G \text{ and } p \in M.$$

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Examples:

- The natural action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$: $\alpha(X, v) := Xv$.
- The action of $\mathrm{SO}_n(\mathbb{R})$ on $S^{n-1} \subset \mathbb{R}^n$: $\alpha(\Theta, x) := \Theta x$.
- The action of SU_n on Her_n: $\alpha(U, H) := UHU^\dagger$.
- The action of $\mathrm{GL}_n(\mathbb{R})$ of Sym_n: $\alpha(X, B) := XBX^\top$.

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A group action is called *transitive* if $\alpha(G, p) = M$ for one $p \in M$ and thus for all $p \in M$.

Induced system. Given a bilinear system $(\widehat{\Sigma})$ on $G \subset \mathrm{GL}_n(\mathbb{R})$ and a group action α of G on M. Then the *induced system* is given by

$$\dot{p}(t) = \mathrm{D}_1\alpha\big(\mathrm{I}, p(t)\big)\left(A + \sum_{k=1}^{m} u_k(t)B_k\right). \tag{Σ_{in}}$$

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Motivation of the above definition:

(Σ_{in}) is “designed” such that $t \mapsto \alpha\big(X(t), p_0\big)$ is a solution of (Σ_{in}) with initial value p_0 if $X(t)$ is a solution of $(\widehat{\Sigma})$ with $X(0) = \mathrm{I}$.

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(Σ_{in}) is "designed" such that $t \mapsto \alpha\big(X(t), p_0\big)$ is a solution of (Σ_{in}) with initial value p_0 if $X(t)$ is a solution of $(\widehat{\Sigma})$ with $X(0) = \mathrm{I}$.

To see this use the identity

$$\alpha(\mathrm{e}^{At}X, p) = \alpha(\mathrm{e}^{At}, \alpha(X, p))$$

to obtain $\mathrm{D}_1\alpha(X, p)AX = \mathrm{D}_1\alpha(I_n, \alpha(Xp))A$ for all $A \in \mathfrak{g}$.

Examples:

- $\alpha(X,v) := Xv$: Induced systems $\mathrm{D}_1\alpha(X,v)\Delta = \Delta v$ and thus $\dot{p}(t) = \Big(A + \sum_{k=1}^{m} u_k(t)B_k\Big)p(t)\,.$
- $\alpha(\Theta,x) := \Theta x$: Induced system $\dot{\varphi}(t) = \Big(A + \sum_{k=1}^{m} u_k(t)B_k\Big)\varphi(t)$
- $\alpha(U,H) := UHU^\dagger$: Induced system $\mathrm{D}_1\alpha(U,H)\Delta = \Delta HU^\dagger + UH\Delta^\dagger$ and thus

$$\dot{P}(t) = \Big[\Big(A + \sum_{k=1}^{m} u_k(t)B_k\Big), P(t)\Big] \qquad \text{(Liouville equation)}$$

- $\alpha(X,B) := XBX^\top$: Induced system $\mathrm{D}_1\alpha(X,B)\Delta = \Delta BX^\top + XB\Delta^\top$ and thus

$$\dot{P}(t) = \Big(A + \sum_{k=1}^{m} u_k(t)B_k\Big)P(t) + P(t)\Big(A + \sum_{k=1}^{m} u_k(t)B_k\Big)^\top \qquad \text{(Lyapunov equation)}$$

Bilinear Control Systems – Advanced Properties

Theorem (Induced System)

Given the bilinear system

$$\dot{X}(t) = \Big(A + \sum_{k=1}^{m} u_k(t) B_k\Big) X(t) \qquad (\widehat{\Sigma})$$

with $A, B_1, ..., B_m \in \mathfrak{g}$ and let $\alpha : G \times M \to M$ be a group action of the respective Lie subgroup G.

(a) Then the reachable sets of (Σ_{in}) are given by $R_{\Sigma_{in}}(p_0) = \alpha\big(R_{\widehat{\Sigma}}(\mathrm{I}_n), p_0\big)$

(b) If G acts transitive on M and $(\widehat{\Sigma})$ is controllable on G then (Σ_{in}) is controllable on M.

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Proof. Follows immediately by the construction of (Σ_{in}).

Note. (b) is not an equivalence.

So far we have seen that any controllability analysis starts with the "computation" of the Lie algebra generated by $A, B_1, \dots, B_m$.

Some "simple" criteria which allow to avoid "painful" computations:

(A) first $\mathrm{SL}_n(\mathbb{R})$

(B) then $\mathrm{SO}(\mathbb{R})$ and SU_n

(C) back to $\mathrm{SL}_n(\mathbb{R})$

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Theorem A [goes back to Silva Leite & P. Crouch]

Given the bilinear system

$$\dot{X}(t) = \Big(A + \sum_{k=1}^{2} u_k(t) B_k\Big) X(t)\,. \qquad (\widehat{\Sigma})$$

with $A, B_1, B_2 \in \mathfrak{sl}_n(\mathbb{R})$.

(a) If A is strongly regular and $u_1 B_1 + u_2 B_2$ satisfies property $(*)$ for some $u_1, u_2 \in \mathbb{R}$ (or vice versa), then $(\widehat{\Sigma})$ is accessible (w.r. to $\mathrm{SL}_n(\mathbb{R})$).

(b) If $u_1 B_1 + u_2 B_2$ is strongly regular for some $u_1, u_2 \in \mathbb{R}$ and $u_1' B_1 + u_2' B_2$ satisfies property $(*)$ for some $u_1', u_2' \in \mathbb{R}$, then $(\widehat{\Sigma})$ is controllable (w.r. to $\mathrm{SL}_n(\mathbb{R})$).

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Strong regularity and $*$-property.

- A is strongly regular, if A is diagonalizable (over $\mathbb{C}$) and all possible eigenvalue differences $\lambda_k - \lambda_l$ for $k \neq l$ are different.
- If A is diagonal then B satisfies the $*$-property (w.r. to A) if $b_{kl} \neq 0$ for all $k \neq l$.

Bilinear Control Systems – Advanced Properties

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with $A, B_1, B_2 \in \mathfrak{sl}_n(\mathbb{R})$.

(a) If A is strongly regular and $u_1 B_1 + u_2 B_2$ satisfies property $(*)$ for some $u_1, u_2 \in \mathbb{R}$ (or vice versa), then $(\widehat{\Sigma})$ is accessible (w.r.t. $\mathrm{SL}_n(\mathbb{R})$).

(b) If $u_1 B_1 + u_2 B_2$ is strongly regular for some $u_1, u_2 \in \mathbb{R}$ and $u_1' B_1 + u_2' B_2$ satisfies property $(*)$ for some $u_1', u_2' \in \mathbb{R}$, then $(\widehat{\Sigma})$ is controllable (w.r.t. $\mathrm{SL}_n(\mathbb{R})$).

Sketch of proof.

- (a) Let A be diagonal and strongly regular and $B := u_1 B_1 + u_2 B_2$. Compute $[A, e_k e_l^\top] = \ldots$ This shows that $e_k e_l^\top$ is an eigenvector of the linear operator $X \mapsto \mathrm{ad}_A(X) := [A, X]$. Next consider the span of $\mathrm{ad}_a(B), \mathrm{ad}_A^2(B), \mathrm{ad}_A^3(B), \ldots$. This yields an ad_A-invariant subspace of $\mathfrak{sl}_n(\mathbb{R})$ which is equal to all $n \times n$-matrices with zero diagonal. This implies $\langle A, B\rangle_{\mathrm{L}} = \mathfrak{sl}_n(\mathbb{R})$
- Part (b) follows from (a) and FT-II, part (c).

Theorem B

Given the bilinear system

$$\dot{U}(t) = \Big(A + u(t)B\Big)U(t)\,. \qquad (\widehat{\Sigma})$$

with $A, B \in \mathfrak{su}_n$. If A is strongly regular and B satisfies $(*)$ (or vice versa), then $(\widehat{\Sigma})$ is controllable (w.r.t. to $\mathrm{SU}_n(\mathbb{R})$).

Sketch of proof. By the same arguments as in the proof of Thm. A we conclude that the complex Lie algebra generated by $\langle A, B\rangle_{\mathrm{L}}$ is equal to $\mathfrak{sl}_n(\mathbb{C})$. Hence the real Lie algebra generated by $\langle A, B\rangle_{\mathrm{L}}$ has to be $\mathfrak{su}_n$ and thus the result follows from FT-II, part (d).

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Remark. There's a similar result for $\mathrm{SO}_n(\mathbb{R})$, but the formulations is a bit more involved.

Theorem C [Jurdjevic, Kupka, Assoudi, Gauthier, ...]

Given the bilinear system

$$\dot{X}(t) = \Big(A + u(t)B\Big)X(t)\,. \qquad (\widehat{\Sigma})$$

with $A, B \in \mathfrak{sl}_n(\mathbb{R})$. Then $(\widehat{\Sigma})$ is controllable (w.r. to $\mathrm{SL}_n(\mathbb{R})$) if –

Theorem C [Jurdjevic, Kupka, Assoudi, Gauthier, ...]

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Reference.

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- R. El Assoudi, J-P.Gauthier, I. Kupka, *On subsemigroups of semisimple Lie groups*, Annales de l'Institut Henri Poincaré, 1996.

Theorem [Finite time controllability]

Given the bilinear system

$$\dot{U}(t) = \Big(A + \sum_{k=1}^{m} u_k(t) B_k\Big) U(t) \qquad (\widehat{\Sigma})$$

on some compact Lie group G with Lie algebra $\mathfrak{g}$ and assume $\langle A, B_1, \ldots, B_m \rangle_L = \mathfrak{g}$.

(a) Then there exist $T > 0$ such that $R_{\widehat{\Sigma}}(\mathrm{I}_n, \leq T) = G$.

(b) If G is additionally simple then there exist $T > 0$ such that even $R_{\widehat{\Sigma}}(\mathrm{I}_n, T) = G$.

Bilinear Control Systems – Advanced Properties

Theorem [Finite time controllability]

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Definition/Sketch of Proof.

- A Lie group is called simple if its Lie algebra has no non-trivial ideals, e.g. SU_n is simple.
- (a) Use $G = \bigcup_{n \in \mathbb{N}} \operatorname{int}\big(R_{\widehat{\Sigma}}(\mathrm{I}_n, \leq n)\big)$ and the compactness of G.
- (b) Use the fact that the zero-time ideal is equal to $\mathfrak{g}$ and thus the zero-time orbit has interior points. Therefore, $R_{\widehat{\Sigma}}(\mathrm{I}_n, T')$ is a neighborhood of the identity for some $T' > 0$ and then repeat the argument of part (a).

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- Agrachev/Sachkov *Control Theory from the Geometric Viewpoint*, Springer, 2004.

Basic Notation & Terminology

- $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ = underlying Hilbert space
- $S(\mathbb{H})$ unit sphere of $\mathbb{H}$
- $\psi, \varphi, \dots \in S(\mathbb{H})$ normalized vectors = pure states
- $\rho = \sum_k \lambda_k \psi_k \psi_k^\dagger$ with $\lambda_k > 0$ and $\sum_k \lambda_k = 1$ density operator/matrix = mixed state
- $D(\mathbb{H})$ = set of all density operators/matrices, , in particular $D_n := D(\mathbb{C}^n)$
- $U(\mathbb{H})$ = the set of all unitary operators on $\mathbb{H}$, in particular $\mathrm{U}_n := U(\mathbb{C}^n)$

Closed Quantum Systems

Schrödinger Equation (= pure state time evolution)

$$(S) \qquad \dot{\psi}(t) = -\mathrm{i}\Big(H_0 + \sum_{k=1}^{m} u_k(t) H_k\Big)\psi(t)$$

with

- drift/system Hamiltionian H_0 (= Hermitian operator on $\mathbb{H}$),
- control Hamiltionians H_k (= Hermitian operators on $\mathbb{H}$) and
- semiclassical control term $\sum_{k=1}^{m} u_k(t) H_k$.

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(S) constitutes obviously a bilinear system on $\mathbb{H} \cong \mathbb{C}^n$ and restricts to $S(\mathbb{C}^n)$.

Closed Quantum Systems

Liouville/von Neumann Equation ($=$ mixed state time evolution)

$$(LvN) \qquad \dot\rho(t) = -\mathrm{i}\Big[H_0 + \sum_{k=1}^{m} u_k(t) H_k, \rho(t)\Big]$$

with

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(LvN) constitutes a bilinear system on the space $\mathrm{Her}_n(\mathbb{C})$
of all Hermitian matrices leaving $D(\mathbb{H})$ invariant

Closed Quantum Systems

Group Lift (= time evolution of the unitary propagator)

$$(P) \qquad \dot{U}(t) = -\mathrm{i}\Big(H_0 + \sum_{k=1}^{m} u_k(t)H_k\Big)U(t), \quad U(0) = \mathrm{I}_n$$

where $U(t) \in \mathrm{U}_n$ (or SU_n) is called the unitary propagator of the system.

Applications to Quantum Control: Closed Quantum Systems

Closed Quantum Systems

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(P) constitutes a bilinear system on Un_n or SU_n.
the previous systems (S) and (LvN) are induced by (P).

Applications to Quantum Control: Closed Quantum Systems

"The" basic controllability result for closed quantum systems:

Controllability of closed QS

Given

$$\dot{\rho}(t) = \Big[\mathrm{i}\Big(H_0 + \sum_{k=1}^{m} u_k(t) H_k\Big), \rho(t)\Big] \quad \rho(0) = \rho_0 \in D_n\,. \qquad (\Sigma_{CQ})$$

and

$$\dot{U}(t) = \mathrm{i}\Big(H_0 + \sum_{k=1}^{m} u_k(t) H_k\Big) U(t) \quad U(0) = \mathrm{I}_n\,. \qquad (\widehat{\Sigma}_{CQ})$$

(a) If $\langle \mathrm{i}H_0, \ldots, \mathrm{i}H_m \rangle_L$ generates a closed subgroup $G \subset \mathrm{U}_n$ then $R_{\widehat{\Sigma}_{CQ}}(\mathrm{I}_n) = G$.

(b) In particular, $\langle \mathrm{i}H_0, \ldots, \mathrm{i}H_m \rangle_L = \mathfrak{u}_n$ (resp. $= \mathfrak{su}_n$) implies $R_{\widehat{\Sigma}_{CQ}}(\mathrm{I}_n) = \mathrm{U}_n$ (resp. $= \mathrm{SU}_n$).

(c) Under the conditions of (b) one has controllability of (Σ_{CQ}) on the unitary orbit of ρ_0.

Proof: Since U_n and SU_n are compact the results follow from FT II and Thm. on induced systems.

Applications to Quantum Control: Closed Quantum Systems

Control of Spin-Systems (Khaneja, Brockett, Schulte-Herbrüggen, Albertini & D'Alessandro, ...)

Given

$$\dot{U}(t) = \mathrm{i}\Big(H_0 + \sum_{k=1}^{m} u_k(t) H_k\Big) U(t) \quad U(0) = \mathrm{I}_n \,. \tag{Σ_{Spin}}$$

If the following two conditions are satisfied (Σ_{Spin}) is controllable on SU_n (resp. Un_n).

- $\langle \mathrm{i}H_1, \ldots, \mathrm{i}H_m \rangle_L$ contains the Lie algebras generated by all local Hamiltonians.
- The interaction Hamiltonian H_0 is given by a sum of two-particle interactions such that the associated connectivity graph is connected.

Proof: Compute commutator in a clever way!

For introducing open systems we need some further concepts from QM:

Tensor products and the tracing out operation

- Given two QS-systems (Σ_1) and (Σ_2) over $\mathbb{H}_1$ and $\mathbb{H}_2$, resp.

  Hilbert space of the coupled system is given by : $\mathbb{H}_1 \otimes \mathbb{H}_2$ ($\mathbb{H}_1 \times \mathbb{H}_2$ too “small”)

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- Given two states $\rho_1 \in D(\mathbb{H}_1)$ and $\rho_2 \in D(\mathbb{H}_2)$. How can we associate to ρ_1 and ρ_2 a state of the coupled system $(\Sigma_1 \otimes \Sigma_2)$: $(\rho_1, \rho_2) \mapsto \rho_1 \otimes \rho_2$ (product state)

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- Given two states $\rho_1 \in D(\mathbb{H}_1)$ and $\rho_2 \in D(\mathbb{H}_2)$. How can we associate to ρ_1 and ρ_2 a state of the coupled system $(\Sigma_1 \otimes \Sigma_2)$: $(\rho_1, \rho_2) \mapsto \rho_1 \otimes \rho_2$ (product state)
- Conversely, given $\rho \in D(\mathbb{H}_1 \otimes \mathbb{H}_2)$ of the coupled system $(\Sigma_1 \otimes \Sigma_2)$. How can one associate to ρ a state of (Σ_1) and (Σ_2), resp.: $\rho \mapsto \operatorname{tr}_{\Sigma_2} \rho$ and $\rho \mapsto \operatorname{tr}_{\Sigma_1} \rho$ (partial trace)

  Defining property: There exists a unique $\Delta \in D(\mathbb{H}_1)$ such that

  $$\operatorname{tr}\big(\rho(A \otimes \mathrm{I})\big) = \operatorname{tr}(ZA) \qquad \text{for all observables } A \text{ of } \Sigma_1$$

  Thus set $\operatorname{tr}_{\Sigma_2} \rho := \Delta$
- Note: $\operatorname{tr}_{\Sigma_2}(\rho_1 \otimes \rho_2) = \rho_1$ and $\operatorname{tr}_{\Sigma_1}(\rho_1 \otimes \rho_2) = \rho_2$. BUT in general $\operatorname{tr}_{\Sigma_2}(\rho) \otimes \operatorname{tr}_{\Sigma_1}(\rho) \neq \rho$.

Quantum channels and complete positivity

- A linear map $\Phi : \mathrm{Her_n} \to \mathrm{Her_n}$ is trace-preserving if $\mathrm{tr}(\Phi(S)) = \mathrm{tr}(S)$ for all $S \in \mathrm{Her_n}$.
- A linear map $\Phi : \mathrm{Her_n} \to \mathrm{Her_n}$ is positive if $\Phi(S) \geq 0$ for all $S \geq 0$.
- A linear map $\Phi : \mathrm{Her_n} \to \mathrm{Her_n}$ is k-positive if $I_k \otimes \Phi : \mathrm{Her_{kn}} \to \mathrm{Her_{kn}}$ is positive.
- A liner map $\Phi : \mathrm{Her_n} \to \mathrm{Her_n}$ is completely positive if Φ is k-positive for all $k \in \mathbb{N}$.
- For $\mathbb{H} = \mathbb{C}^n$ one has the equivalence:

$$\Phi \text{ completely positive} \iff \Phi\ n\text{-positive} \iff \text{Choi-matrix positive}$$

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- QM viewpoint: These are the right objects to describe state transitions in open QM
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Examples and characterizations (for $\mathbb{H} = \mathbb{C}^n$):

A linear map $\Phi : \mathrm{TrC}(\mathbb{H}) \to \mathrm{TrC}(\mathbb{H})$ is a quantum channel if and only if one of the following representations holds:

$$\Phi(\rho) = \sum_{k=1}^{N} V_k \rho V_k^\dagger \quad \text{with} \quad \sum_{k=1}^{N} V_k^\dagger V_k = \mathrm{I}_n \qquad \text{(Kraus)}$$

or

$$\Phi(\rho) = \mathrm{tr}_{\Sigma_B}\left(U(\rho \otimes \omega)U^\dagger\right) \quad \text{with} \quad \omega \in D(\mathbb{H}_B) \quad \text{and} \quad U \in U(\mathbb{C}^n \otimes \mathbb{H}_B)\,. \qquad \text{(Stinespring)}$$

- Kraus for $N = 1$: $\Phi(\rho) = U\rho U^\dagger$ with U unitary.
- Generalizations to $\dim \mathbb{H} = \infty$ do exist!

The dynamics of open quantum systems

Postulate: The dynamics of an open quantum systems (Σ) is described by a one-parameter semigroup of quantum channels, i.e. if ρ_0 is the initial state of (Σ) the time evolution is given by

$$\rho(t) = \mathrm{e}^{tL}\rho_0 \,,$$

or, alternatively, by the solution of the linear "ODE"

$$\dot{\rho}(t) = L\rho(t)\,, \quad \rho(0) = \rho_0 \,,$$

such that e^{Lt} is a quantum channel for all $t \geq 0$.

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Problem: How to choose L such that e^{Lt} is a Q-channel for all $t \geq 0$?

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Theorem (Gorini, Kossakowski, Sudarshan / Lindblad 1976)

$L : \mathrm{Her_n} \to \mathrm{Her_n}$ is the infinitesimal generator of a (uniformly continuous) semigroup of quantum channels if and only if it allows the following representation:

$$L(\rho) = \mathrm{i}ad_H(\rho) + \sum_k \Big(2V_k\rho V_k^\dagger - V_k^\dagger V_k\rho - \rho V_k^\dagger V_k\Big) \qquad \text{(GKSL)}$$

Remark:

- Gorini, Kossakowski, Sudarshan treated the finite dimensional case.
- Lindblad the uniformly continuous infinite dimensional case – proof rather involved!
- The infinite dimensional strongly continuous case is still open.

Control Problem: Given a controlled GKSL-equation

$$\dot{\rho}(t) = \mathrm{i}\Big(ad_{H_0}(\rho(t)) + \sum_{k=1}^{m} u_k(t) ad_{H_k}(\rho(t))\Big) + \gamma \sum_k \Big(2 V_k \rho(t) V_k^\dagger - V_k^\dagger V_k \rho(t) - \rho(t) V_k^\dagger V_k\Big), \gamma \geq 0 \tag{c-GKSL}$$

What can be said about its reachable sets?

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Theorem (Ando 1989 / vom Ende, D., Keyl, Schulte-Herbrüggen 2019)

Assume that (c-GKSL) is unital, i.e. that I_n is an equilibrium of (c-GKSL).

(a) For $\gamma > 0$ fixed, the reachable set $R_{\text{c-GKSL}}(\rho_0)$ is contained in the set of all $\rho \in D_n$ which are majorized by ρ_0.

(b) If $\gamma \in \{0, 1\}$ can be switch on and off and if the following conditions hold:

  (i) $\langle \mathrm{i}H_0, \ldots, \mathrm{i}H_m \rangle_{LA} \supset \mathfrak{su}_n$

  (ii) $V_1 = V_1^\dagger \neq 0$ and $V_i = 0$ for $i \geq 2$

  then the reachable set $R_{\text{c-GKSL}}(\rho_0)$ coincides with the set of all $\rho \in D_n$ which are majorized by ρ_0.

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- Optimal control
- Infinite dimensional quantum systems
- “Ensemble” control of quantum systems
- Quantum speed limits
- ...
- ...

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- Optimal control (D. Sugny, ...)
- Infinite dimensional quantum systems (N. Boussaid, Th. Chambrion, E. Pozzoli, P. Rouchon, , M. Mirrahimi, U. Boscain, M. Sigalotti, ...)
- “Ensemble” control of quantum systems (U. Boscain, M. Sigalotti, ...)
- Quantum speed limits (Th. Chambrion, E. Pozzzoli, K. Beauchard, ...)
- ...
- ...

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That’s it!

Thanks a lot for your attention!

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