Ensemble control of n-level quantum systems with a scalar control

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Ensemble control

Let us consider a bilinear system depending on a parameter s of the form

$$\frac{\mathrm{d}X(t,\theta)}{\mathrm{d}t} = \Big(A(\theta) + \sum_{i=1}^{m} u_i(t)B_i(\theta)\Big)X(t,\theta).$$

where $A(\theta), B_i(\theta)$ are $n \times n$ matrices determined by a parameter θ and $u(\cdot) = (u_1, \ldots, u_m)$ is the time-dependent control that takes values in $\mathbb{R}^m$. This modelizes the problem of

- controlling a family of systems with the same control;
- controlling a system with some unknown parameter.

Remark

For the ensemble control of n-level quantum systems, the state $X(\cdot,\theta)$ evolves on the Lie group $U(n)$ or $SU(n)$. Consequently $A(\theta)$ and $B_i(\theta)$ are skew-Hermitian or traceless skew-Hermitian matrices.

Ensemble control

The goals of ensemble control can be:

- Steer a continuum of systems between states of interest with the same control input (i.e. eigenstates in quantum systems);
- Steer an initial distribution of the ensemble to a final distribution with the same control (i.e. from initial states $\psi_0(\theta)$ to final states $\psi_1(\theta)$);
- Stabilize the ensemble to the common final state;
- Optimal ensemble control e.t.c

Ensemble control of finite family of systems

Consider a finite family of control systems with states X_k, $k = 1, \dots, K$ on connected, simple and compact Lie groups G_k (i.e. $SU(n)$, $SO(n)\dots$):

$$\frac{\mathrm{d}}{\mathrm{d}t} X_k(t) = (A_k + u(t) B_k) X_k(t).$$

If each system is controllable, and for all $k, l \in \{1, \dots, K\}$,$k \neq l$, there doesn't exist a Lie algebra isomorphism $f : \mathfrak{g}_k \to \mathfrak{g}_l$ such that $f(A_k) = A_l$ and $f(B_k) = B_l$. Then the finite ensemble of systems are simultaneously controllable (see [Belhadj, Salomon and Turinici, 2015]).

Ensemble (simultaneous) controllability

Ensemble controllability means that for all $(X_0^1, \dots X_0^K) \in G_1 \times \cdots \times G_K$ and $(X_1^1, \dots X_1^K) \in G_1 \times \cdots \times G_K$, there exists a control $u : [0, T] \to \mathbb{R}$ such that for all $k \in \{1, \dots, K\}$, the solution $X_k(\cdot)$ associated with $u(\cdot)$ satisfies

$$X_k(0) = X_0^k, \qquad X_k(T) = X_1^k.$$

Bloch equations driven by two inputs

Consider the following family of control systems indexed by ω and ϵ on $SO(3)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t,\omega,\epsilon) = \begin{pmatrix} 0 & -\omega & \epsilon u(t) \\ \omega & 0 & -\epsilon v(t) \\ -\epsilon u(t) & \epsilon v(t) & 0 \end{pmatrix} \cdot M(t,\omega,\epsilon), \quad M(0,\omega,\epsilon) = I.$$

with $\omega \in [a,b]$ (Larmor dispersion), $\epsilon \in [1-\delta, 1+\delta]$ (rf inhomogeneity).

Ensemble controllability

For any continuous function $\Theta : [a,b] \times [1-\delta, 1+\delta] \to SO(3)$, we can steer the ensemble from the common initial state $M(0,\omega,\epsilon) = I$ arbitrarily close to the continuously parameterized target states $\Theta(\omega,\epsilon)$ with bounded controls $u(\cdot)$ and $v(\cdot)$ (see [Li and Khaneja, 2009] and [Beauchard, Coron and Rouchon 2009]).

2-level quantum systems driven by complex-valued control

The same result holds for the following systems on $SU(2)$ driven by $u : [0, T] \to \mathbb{C}$

$$i\frac{\mathrm{d}}{\mathrm{d}t}X(t,\omega,\epsilon) = \begin{pmatrix} \omega & \epsilon u(t) \\ \epsilon \bar{u}(t) & -\omega \end{pmatrix} X(t,\omega,\epsilon) \quad X(0,\omega,\epsilon) = I.$$

Uniform population inversion

For $U(\omega,\epsilon) \in SU(2)$ indexed by ω and ϵ, it realizes a uniform population inversion if for all $(\omega,\epsilon) \in [a,b] \times [1-\delta, 1+\delta]$, there exists $\beta \in [0, 2\pi)$ such that

$$U(\omega,\epsilon)\mathbf{e}_1 = \exp(i\beta)\mathbf{e}_2.$$

A crucial operation in the proof of ensemble controllability is to find a control law $u(\cdot)$ such that at the final instant $t = T$, the family $X(T,\omega,\epsilon)$ realizes uniform population inversion. With multiple degrees of freedom in the control (i.e. $u(\cdot) \in \mathbb{R}^n$ with $n > 1$ or $u(\cdot) \in \mathbb{C}$), this can be approximately realized by using an adiabatic following.

Ensemble control with scalar control

Let us replace the complex control in a two-level system by a scalar control $u(\cdot) \in \mathbb{R}$,

$$i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \begin{pmatrix} \omega & \delta u(t) \\ \delta u(t) & -\omega \end{pmatrix} \psi(t), \tag{1}$$

Adiabatic approximation is no longer valid with only one degree of freedom in the control. A standard method to generate multiple degrees of freedom in quantum control is the rotating wave approximation.

Combination of RWA and AA for two-level systems [Robin, Augier, Boscain, Sigalotti, 2022]

Consider two time scales $\epsilon_1, \epsilon_2 > 0$ and a control law of the type

$$u_{\epsilon_1,\epsilon_2}(t) = 2\epsilon_1 u(\epsilon_1\epsilon_2 t)\cos\left(2Et + \frac{\Delta(\epsilon_1\epsilon_2 t)}{\epsilon_1\epsilon_2}\right). \tag{2}$$

Here u, Δ are real-valued smooth functions on [0,T].

Extension to n-level system

Let us consider a continuum of n-level systems described by the Schrödinger equation

$$i\dot{\psi}(t) = (H(\alpha) + \omega(t)H_c(\delta))\psi(t), \tag{3}$$

where $\omega(\cdot)$ is a real-valued control. Here we assume $H(\alpha)$ is determined by an unknown parameter α in a closed and connected domain $\mathcal{D}$ of $\mathbb{R}^m$ and has the following structure

$$H(\alpha) = \begin{pmatrix} \lambda_1(\alpha) & 0 & \dots & 0 \\ 0 & \lambda_2(\alpha) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n(\alpha) \end{pmatrix},$$

where $\lambda_j : \mathcal{D} \to \mathbb{R}$ is a continuous function for each $j \in \{1, \dots, n\}$.

Extension to n-level system

$H_c(\delta)$ is a symmetric matrix that describes the control coupling between the eigenstates of the system and has the form

$$H_c(\delta) = \begin{pmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{12} & \delta_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \delta_{n-1,n} \\ \delta_{1n} & \cdots & \delta_{n-1,n} & \delta_{n,n} \end{pmatrix},$$

We assume that each δ_{jk} is unknown but that it belongs to some closed interval $\mathcal{I}_{jk} = \left[\delta_{jk}^0, \delta_{jk}^1\right]$ in $\mathbb{R}$.

Extension to n-level system

Goal

For $p, q \in \{1, \dots, n\}$ s.t. $p \neq q$, one would like to find a uniform control $\omega(\cdot)$ for the family of quantum systems s.t. if at $t = 0$, all systems have the same initial state $\psi(0) = \mathbf{e}_p$, then at the final instant all systems are close to final states of form $e^{i\theta}\mathbf{e}_q$ for some $\theta \in \mathbb{R}$.

We consider the control law

$$\omega_{\epsilon_1\epsilon_2}(t) = 2\epsilon_1 u(\epsilon_1\epsilon_2 t) \cos\left(\int_0^t f(\epsilon_1\epsilon_2\tau)\mathrm{d}\tau\right), \tag{4}$$

where $u, f : [0, T] \to \mathbb{R}$ are functions to be chosen.

Theorem

Let us assume that for all $1 \leq j < k \leq n$, and for all $\alpha \in \mathcal{D}$, $\lambda_k(\alpha) - \lambda_j(\alpha) > 0$. Fix $1 \leq p < q \leq n$. Assume that δ_{pq} belongs to a closed interval $\mathcal{I}_{pq} = \left[\delta_{pq}^0, \delta_{pq}^1\right]$ such that $0 \notin \mathcal{I}_{pq}$ and there exist $0 < v_0 < v_1$ such that

1. For all $\alpha \in \mathcal{D}$, $\lambda_q(\alpha) - \lambda_p(\alpha) \in (v_0, v_1)$;
2. For all $1 \leq j < k \leq n$ such that $(j,k) \neq (p,q)$ and all $\alpha \in \mathcal{D}$, we have $\lambda_k(\alpha) - \lambda_j(\alpha) \notin [v_0, v_1]$.

Fix $T > 0$ and take $u, f \in \mathcal{C}^2([0,T], \mathbb{R})$ such that

i) $(u(0), f(0)) = (0, v_0)$ and $(u(T), f(T)) = (0, v_1)$;

ii) $\forall s \in (0,T), u(s) > 0$ and $\forall s \in [0,T], \dot{f}(s) > 0$.

Denote by $\psi_{\epsilon_1,\epsilon_2}$ the solution of (3) with initial condition $\psi_{\epsilon_1,\epsilon_2}(0) = \mathbf{e}_p$ and the control law $\omega_{\epsilon_1,\epsilon_2}$ as in (4). Then there exist $C > 0$ and $\eta > 0$ such that for every $\alpha \in \mathcal{D}$ and every $(\epsilon_1, \epsilon_2) \in (0, \eta)^2$,

$$\left\| \psi_{\epsilon_1,\epsilon_2}\left(\frac{T}{\epsilon_1\epsilon_2}\right) - \exp(i\theta)\mathbf{e}_q \right\| \leq C(\epsilon_2\epsilon_1^{-1} + \epsilon_1^{3/2}\epsilon_2^{-1/2} + \epsilon_1 + \epsilon_1^{5/2}\epsilon_2^{-3/2}),$$

for some $\theta \in \mathbb{R}$.

Idea of the proof

For $E \in \mathbb{R}$ and $1 \leq j \leq k \leq n$, let us define

$$A_{jk}(E) = \begin{cases} \exp(iE)\mathbf{e}_{jk} + \exp(-iE)\mathbf{e}_{kj} & \text{if } j < k, \\ \cos(E)\mathbf{e}_{jj} & \text{if } j = k. \end{cases}$$

Let us use the control signal $\omega_{\epsilon_1\epsilon_2}(t) = 2\epsilon_1 u(\epsilon_1\epsilon_2 t)\cos\left(\int_0^t f(\epsilon_1\epsilon_2\tau)\mathrm{d}\tau\right)$ and recast the system in the interaction frame $\psi(t) = \exp(-itH(\alpha))\psi_I(t)$. The new dynamics are characterized by the Hamiltonian

$$H_I(t) = \sum_{j=1}^{n-1}\sum_{k=j+1}^{n} \epsilon_1\delta_{jk}u(\epsilon_1\epsilon_2 t)\left[A_{jk}(\phi_{jk}^1(t)) + A_{jk}(\phi_{jk}^{-1}(t))\right].$$

where

$$\phi_{jk}^{\sigma}(t) = (\lambda_j - \lambda_k)t + \sigma\int_0^t f(\epsilon_1\epsilon_2\tau)\mathrm{d}\tau. \qquad (5)$$

Idea of the proof

$$H_I(t) = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} \epsilon_1 \delta_{jk} u(\epsilon_1 \epsilon_2 t) \left[A_{jk}\left(\phi_{jk}^1(t)\right) + A_{jk}\left(\phi_{jk}^{-1}(t)\right) \right].$$

Notice that with the Hypothesis of the Theorem, for all phases $\phi_{jk}^{\sigma}(t)$ except $\phi_{pq}^1(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_{jk}^{\sigma}(t) = \lambda_j(\alpha) - \lambda_k(\alpha) + \sigma f(\epsilon_1 \epsilon_2 t) \neq 0$$

Because these frequencies are nowhere vanishing, we can eliminate all oscillatory terms in $H_I(t)$ except that with the phase $\phi_{pq}^1(t)$. We then project the system onto the two-dimensional space spanned by $\{\mathbf{e}_p, \mathbf{e}_q\}$. The resulting dynamics can be approximated by the following effective Hamiltonian, which is driven by a complex-valued control:

$$H_{\text{eff}}(t) = \begin{pmatrix} \lambda_p(\alpha) & \epsilon_1 u(\epsilon_1 \epsilon_2 t) \exp(i \int_0^t f(\epsilon_1 \epsilon_2 \tau) \mathrm{d}\tau) \\ \epsilon_1 u(\epsilon_1 \epsilon_2 t) \exp(-i \int_0^t f(\epsilon_1 \epsilon_2 \tau) \mathrm{d}\tau) & \lambda_q(\alpha) \end{pmatrix}$$

Example

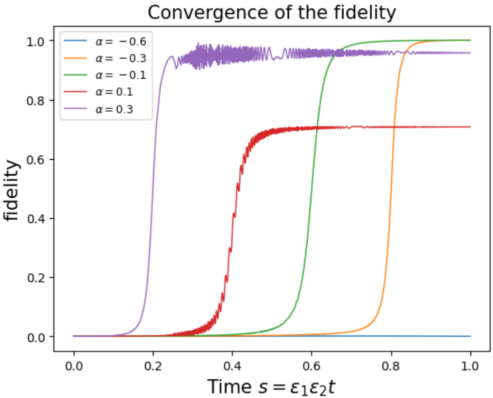
Consider the four-level system with drift and control Hamiltonians

$$H(\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1+\alpha & 0 & 0 \\ 0 & 0 & 3+2\alpha & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad H_c = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Fix the initial state $\psi_{\epsilon_1,\epsilon_2}(0) = (0,0,1,0)^\top$. In order to realize a population inversion between $(0,0,1,0)^\top$ and $(0,0,0,1)^\top$, let us fix $T = 1$, $v_0 = 3$ and $v_1 = 5$. We use the time scale $(\epsilon_1, \epsilon_2) = (10^{-5/3}, 10^{-7/3})$ and test the sharpness of conditions for $\alpha \in \{-0.6, -0.3, -0.1, 0.1, 0.3\}$.

- $\alpha = -0.3, -0.1$: all conditions of the Theorem are satisfied.
- $\alpha = -0.6$: $\lambda_4(\alpha) - \lambda_3(\alpha) \notin (v_0, v_1)$. Hypothesis 1 of the Theorem is violated.
- $\alpha = 0.1, 0.3$: $\lambda_3(\alpha) - \lambda_1(\alpha) \in [v_0, v_1]$. Hypothesis 2 of the Theorem is violated.

Example

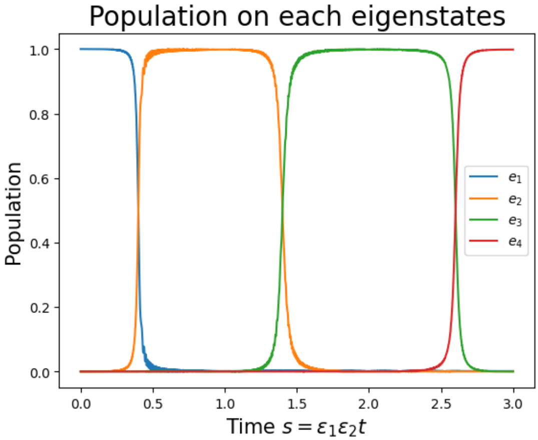


For $s \in [0,1]$, let us define the fidelity of the inversion as the population on the target state $\mathbf{e}_q$ at instant $t = s/(\epsilon_1 \epsilon_2)$:

$$\text{fid}(s) = \left| \left\langle \psi_{\epsilon_1,\epsilon_2}\left(\frac{1}{\epsilon_1 \epsilon_2}\right), \mathbf{e}_q \right\rangle \right|^2 .$$

Example

Notice that a direct inversion between $\mathbf{e}_1$ and $\mathbf{e}_4$ is impossible since $\delta_{14} = 0$. But it is possible to realize a complete population transition between $\mathbf{e}_1$ and $\mathbf{e}_4$ by successive inversions between $(\mathbf{e}_1, \mathbf{e}_2)$, between $(\mathbf{e}_2, \mathbf{e}_3)$, and between $(\mathbf{e}_3, \mathbf{e}_4)$.



Another result

Assume that the assumptions of Theorem are satisfied, and moreover, for all $1 \leq j < k \leq n$ and $\alpha \in \mathcal{D}$, $\lambda_k(\alpha) - \lambda_j(\alpha) \notin [2v_0, 2v_1]$. Then there exist $C > 0$ and $\eta > 0$ such that for every $(\epsilon_1, \epsilon_2) \in (0, \eta)^2$,

$$\min_{\theta \in [0,2\pi]} \left\| \psi_{\epsilon_1,\epsilon_2}\left(\frac{T}{\epsilon_1 \epsilon_2}\right) - \exp(i\theta) e_q \right\| \leq C \max\left(\frac{\epsilon_1^2}{\epsilon_2}, \frac{\epsilon_2}{\epsilon_1}\right).$$

Thank you for your attention!

References

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