Control of nonlinear quantum systems

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Outline

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1 Background

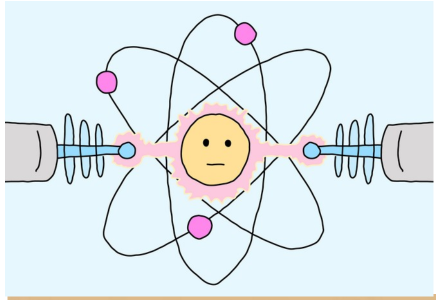
2 bust control in single qubit systems

3 Robust control in nonlinear systems

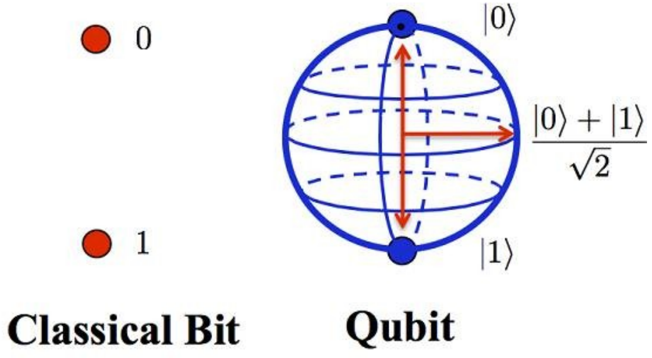
4 Conclusion

Background

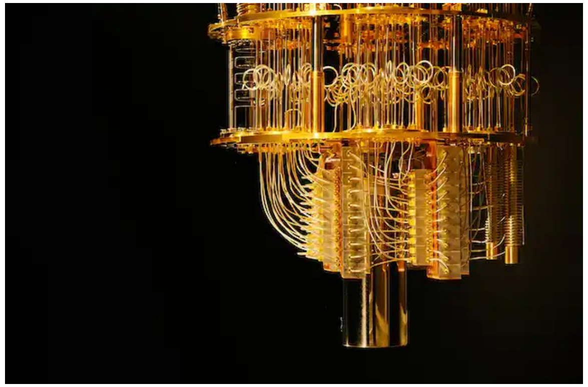
Classic picture ← "control" → Quantum picture

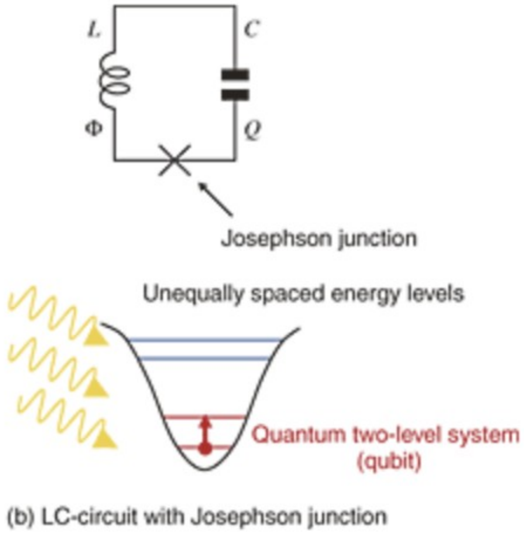


Background

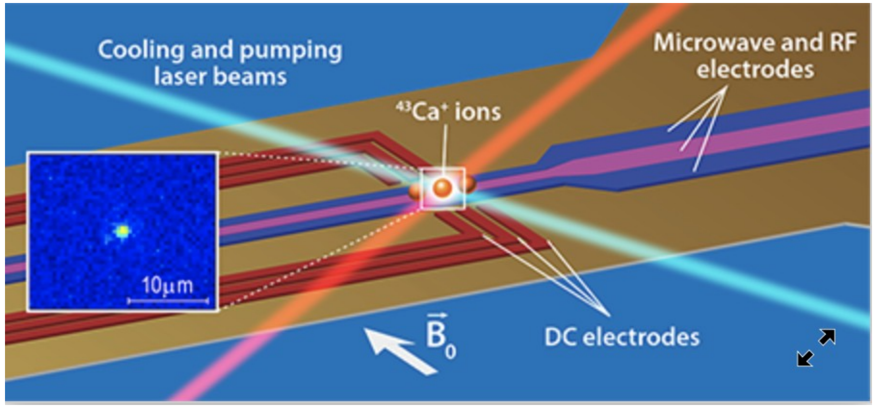


superconducting circuits:





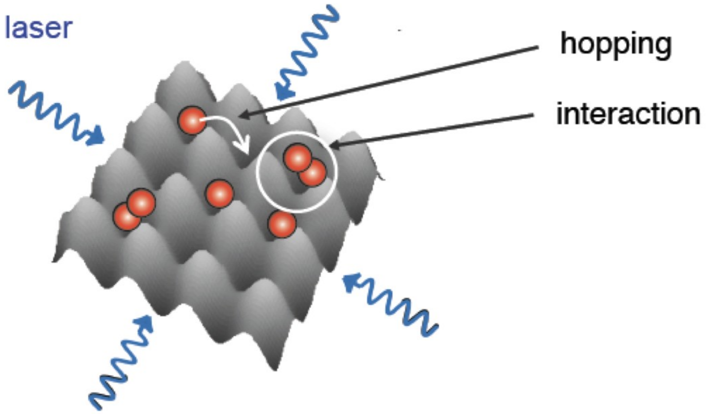
trapped ions:



- Decoherence
- Non-robustness

…

BECs:



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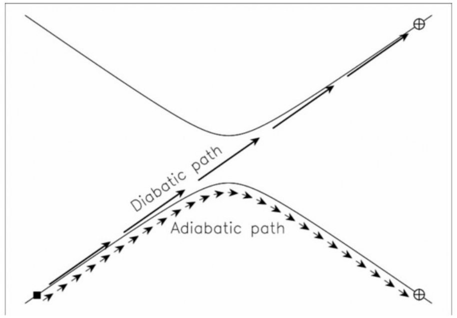
Background

Quantum control → Transfer initial state to a target state with desired properties

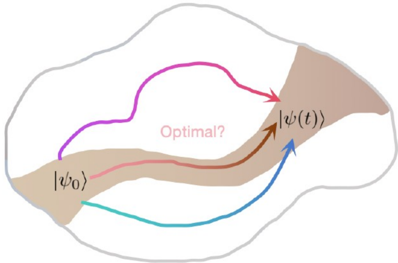
Quantum control is playing a central role in physics and chemistry and has significant development both in linear and non-linear systems to speed up the dynamics, to minimize the effect from decoherence, achieve high-fidelity and good robustness against quantum fluctuations.

How to realize these goals technically?

Adiabatic Passage (AP):

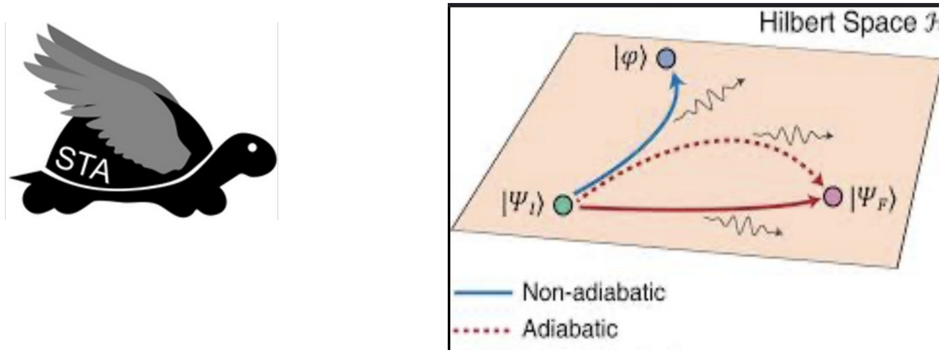


Quantum optimal control (QOC):

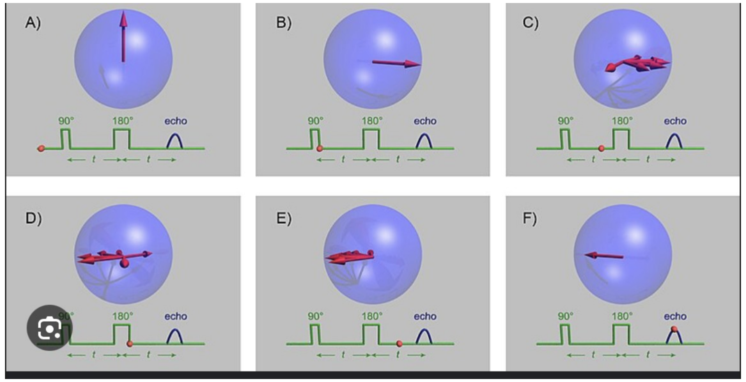


$$J(x) = \int_{t_i}^{t_f} L_0(x(t), \dot{x}(t), t)dt.$$

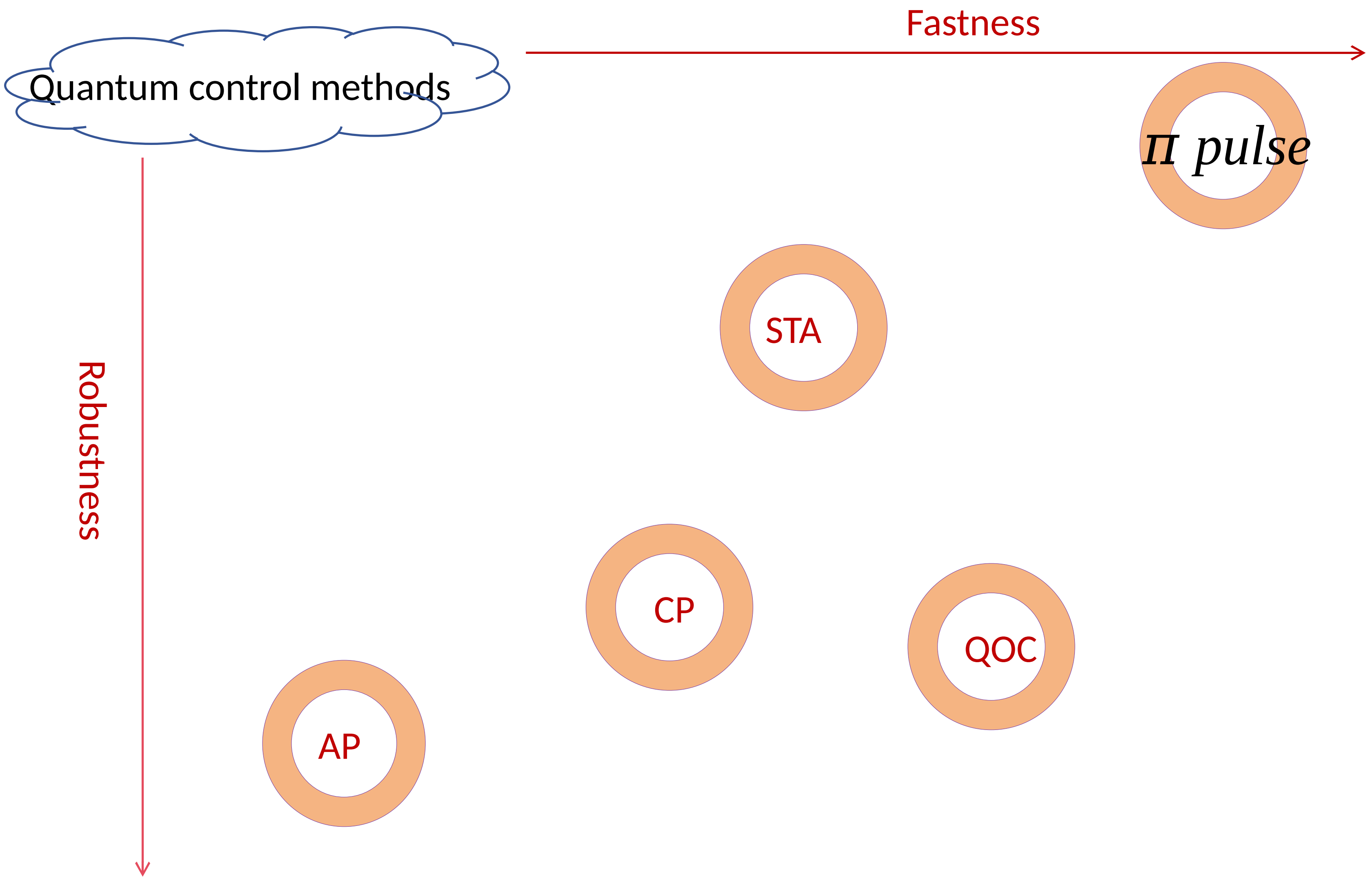
Shortcut to adiabaticity (STA):



Composite pulses:



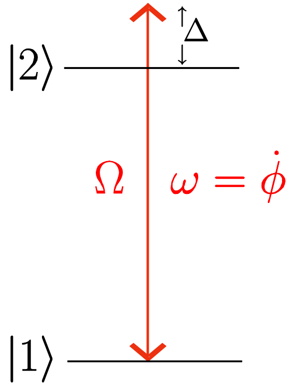
Background



AP: Adiabatic passage
CP: Composite pulses
STA: Shortscuts to adiabaticity
QOC: quantum optimal control

Outline

Single qubit system



$$\mathrm{H}^{[\Omega(t),\Delta(t)]} = \frac{1}{2}\begin{bmatrix} -\Delta(t) & \Omega(t) \\ \Omega(t) & \Delta(t) \end{bmatrix}$$

Minimum time for *complete population transfer*:
Given by the minimum area: Ω-π for Δ =0, i.e. “π -pulse”

To satisfy specific constraints, we require larger pulse area
For instance: robustness

$$H_{\alpha,\delta}(t) = \frac{\hbar}{2}\begin{bmatrix} -\Delta(t) & \Omega(t) \\ \Omega(t) & \Delta(t) \end{bmatrix} + \underbrace{\frac{\hbar}{2}\begin{bmatrix} -\delta & \alpha\Omega(t) \\ \alpha\Omega(t) & \delta \end{bmatrix}}_{\text{perturbation}}$$

Solution without perturbation:

$$\phi(t) = \begin{bmatrix} e^{i\varphi/2}\cos(\theta/2) \\ e^{-i\varphi/2}\sin(\theta/2) \end{bmatrix} e^{-i\gamma/2} \longrightarrow \text{TDSE}$$

$$\begin{cases} \dot{\theta} = \Omega \sin\varphi \\ \dot{\varphi} = \Delta + \Omega\cos\varphi \,\mathrm{cotan}\,\theta \\ \dot{\gamma} = \Omega\dfrac{\cos\varphi}{\sin\theta} = \dot{\theta}\dfrac{\mathrm{cotan}\,\varphi}{\sin\theta} \end{cases}$$

Final target: $\phi_T = \phi(t_f)$

Single qubit system

Inverse engineering (tracking) solution: infinitely many solutions for θ (t), φ (t), γ (t) that lead to the final target, but does not guarantee the robust process

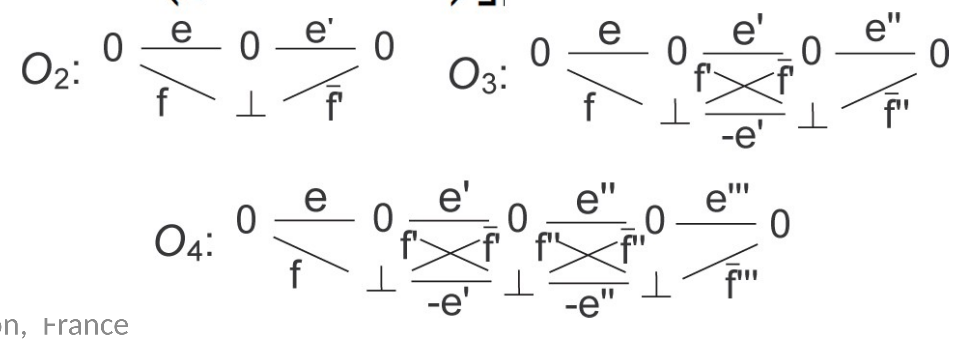
Perturbative series of the total solution $\phi_{\alpha,\delta}(t)$ around the target (at final time)

$$|\langle\phi_T|\phi_{\alpha,\delta}(t_f)\rangle|^2 = 1 + (O_1 + \bar{O}_1) + (O_2 + \bar{O}_2 + \bar{O}_1 O_1) + (O_3 + \bar{O}_3 + \bar{O}_1 O_2 + O_1 \bar{O}_2) + \cdots$$

We choose the parameters θ (t), φ (t), γ (t) such that $O_{n\leq N}=0$: Self-corrective solution at order N !

$$O_2 + \bar{O}_2 + \bar{O}_1 O_1 = -\left|\int_{t_i}^{t_f} f(t)dt\right|^2 \qquad f = \langle\phi_0|V|\phi_\perp\rangle = \frac{1}{2}\left[\delta\,\sin\theta + \alpha\left(\frac{1}{2}\dot{\gamma}\,\sin2\theta - i\dot{\theta}\right)\right]e^{i\gamma},$$

Diagrammatic construction of the perturbative series at any order



Single qubit system

RIO: Target state = excited state/ superposed state + optimization

By reformulating the external driving expressions:

$$\Delta = \dot{\varphi} - \dot{\gamma}\cos\theta,$$

$$\Omega = \sqrt{\dot{\theta}^2 + \dot{\gamma}^2 \sin^2\theta} = |\dot{\gamma}|\sqrt{(\dot{\tilde{\theta}})^2 + \sin^2\tilde{\theta}}$$

(Pulse area)

$$\mathcal{A} = \int_{t_i}^{t_f} dt\,\Omega(t) = \int_{\gamma_i}^{\gamma_f} d\gamma\sqrt{(\dot{\tilde{\theta}})^2 + \sin^2\tilde{\theta}},$$

By construting the seconde-order perturbation $\widetilde{O}_2 = 0 \quad \delta = 0$

$$\psi_1(\tilde{\theta}) = -\frac{1}{4}\int_{\gamma_i}^{\gamma_f} d\gamma(\sin 2\tilde{\theta} - 2\tilde{\theta})\sin\gamma \equiv \int_{\gamma_i}^{\gamma_f} d\gamma\,\varphi_1(\gamma,\tilde{\theta})$$

$$= -\frac{1}{2}(\theta_f\cos\gamma_f - \theta_i\cos\gamma_i), \tag{11a}$$

$$\psi_2(\tilde{\theta}) = \frac{1}{4}\int_{\gamma_i}^{\gamma_f} d\gamma(\sin 2\tilde{\theta} - 2\tilde{\theta})\cos\gamma \equiv \int_{\gamma_i}^{\gamma_f} d\gamma\,\varphi_2(\gamma,\tilde{\theta})$$

$$= \frac{1}{2}(\theta_i\sin\gamma_i - \theta_f\sin\gamma_f) \tag{11b}$$

Applying the Euler-Lagrange optimization,

$$\mathrm{grad}\,\mathcal{A}(\tilde{\theta}) + \lambda_1\,\mathrm{grad}\,\psi_1(\tilde{\theta}) + \lambda_2\,\mathrm{grad}\,\psi_2(\tilde{\theta}) = 0,$$

The optimal robust trajectory is given

$$\ddot{\tilde{\theta}} = 2(\dot{\tilde{\theta}})^2\,\mathrm{cotan}\,\tilde{\theta} + \sin\tilde{\theta}\cos\tilde{\theta}$$
$$+ (\lambda_1\sin\gamma - \lambda_2\cos\gamma)\left((\dot{\tilde{\theta}})^2 + \sin^2\tilde{\theta}\right)^{3/2}.$$

$$T_{\min} = \frac{1}{\Omega_0}\int_{\gamma_i}^{\gamma_f} d\gamma\sqrt{(\dot{\tilde{\theta}})^2 + \sin^2\tilde{\theta}}.$$

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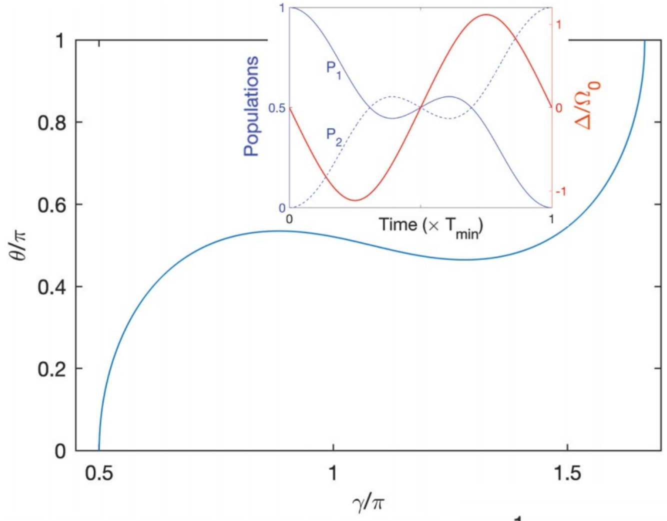
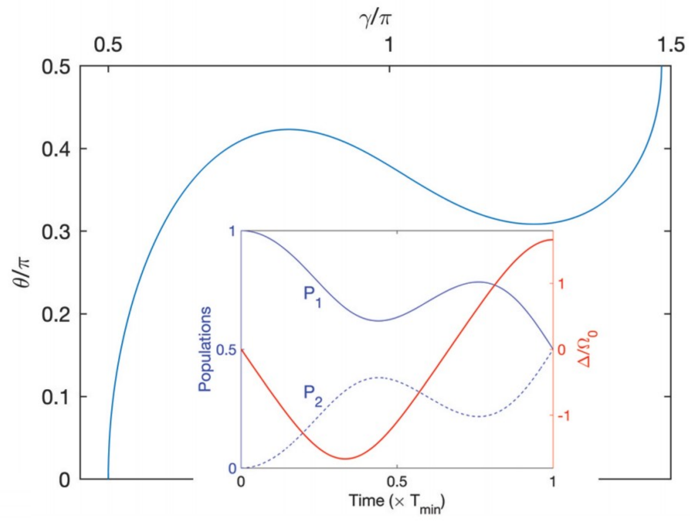
Single qubit system

Complete population transfer



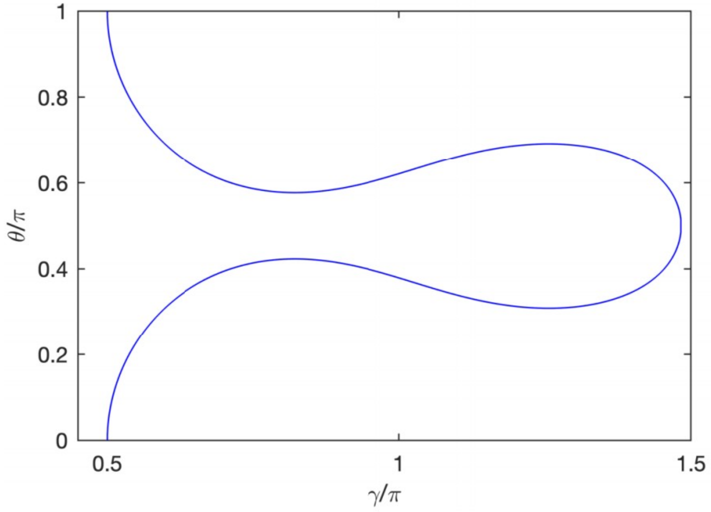
$\sim 1.86\pi$

Half population transfer



$\sim 1.29\pi$

NOT gate



$\sim 2.58\pi$

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2. Robust control in single qubit systems
3. Robust control in nonlinear systems
4. Conclusion

Nonlinear systems

State transfer of interacting BECs in the optical alttice:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\left(\hbar\frac{\partial}{\partial x}\right)^2\psi + V_0\cos(2k_L x - \varphi(t))\psi + \frac{4\pi\hbar^2 a_s}{m}|\psi|^2\psi,$$

Dimensionless: $E_R = \hbar\omega_R = \hbar^2 k_L^2/2m$

$$\tilde{x} = k_L x, \quad \phi = \frac{\psi}{\sqrt{n_0}}, \quad \tilde{t} = \frac{E_R}{\hbar}t,$$

$$\tilde{V}_0 = \frac{V_0}{E_R}, \quad \tilde{\alpha} = \frac{m}{E_R k_L}\alpha(t), \quad g = \frac{8 n_0 \pi a_s}{k_L^2},$$

After transformation:

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\left(\frac{\partial}{\partial x} - iA(t)\right)^2\phi + V_0\cos(2x)\phi + g|\phi|^2\phi,$$

$$\varphi = \int A(s)ds$$

Two-mode approximation

$$\phi(x,t) = c_1(t)|2\hbar k_L\rangle + c_0(t)|0\hbar k_L\rangle,$$

$$i\frac{\partial c_0}{\partial t} = \frac{1}{2}(0 - A(t))^2 c_0 + \frac{V_0}{2}c_1 + g(1 + |c_1|^2)c_0,$$

$$i\frac{\partial c_1}{\partial t} = \frac{1}{2}(2k_L - A(t))^2 c_1 + \frac{V_0}{2}c_0 + g(1 + |c_0|^2)c_1,$$

$\varphi = \int(\Delta(s) + 2)/2ds$ $\quad$ $b_j = c_j e^{-i\int A^2(t)/2dt}$, $\quad$ $V_0 = \Omega_0 E_R$

Nonlinear driven two-mode system: $\quad |b_1|^2 + |b_2|^2 = 1$

$$i\begin{bmatrix}\dot{b}_1\\ \dot{b}_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\Delta + g(|b_2|^2 - |b_1|^2) & \Omega\\ \Omega & -\Delta - g(|b_2|^2 - |b_1|^2)\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix},$$

- Nonintegrability
- Instability

↓

- ✖ Perturbation thoery
- ✖ Adiabatic condition

Nonlinear systems

Hamiltonian in the phase space (classical and quantum correspondance) :

$$i\frac{db_1}{dt} = \frac{\partial h}{\partial b_1^*}, \quad i\frac{db_2}{dt} = \frac{\partial h}{\partial b_2^*}$$

$$h = \frac{\Delta}{2}(b_1^* b_1 - b_2^* b_2) + \frac{g}{2}|b_1|^2|b_2|^2 - \frac{g}{4}\left(|b_1|^4 + |b_2|^4\right) + \frac{\Omega}{2}b_1^* b_2 + \frac{\Omega}{2}b_2^* b_1,$$

canonical variables: (I, φ)

$$b_j = \sqrt{I_j}e^{i\varphi_j}$$

$$I = |b_2|^2 - |b_1|^2$$

$$\varphi = \varphi_2 - \varphi_1$$

$$h = -\frac{\Delta}{2}I - \frac{g}{4}I^2 + \frac{\Omega}{2}\sqrt{1 - I^2}\cos\varphi,$$

$$\dot{I} = -\partial h/\partial\varphi \quad \dot{\varphi} = \partial h/\partial I$$

$$\dot{I} = \frac{\Omega}{2}\sqrt{1 - I^2}\sin\varphi,$$

$$\dot{\varphi} = -\frac{\Delta}{2} - \frac{g}{2}I - \frac{\Omega I}{2\sqrt{1 - I^2}}\cos\varphi.$$

non-rigid pendulum

Stationary states (elliptic fixed points) :

$$\dot{I} = 0, \ \dot{\varphi} = 0$$

$$\varphi = 0, \pi \quad \Delta = -gI \pm \frac{\Omega I}{\sqrt{1 - I^2}},$$

$$\Delta = 0$$

resonance

$$\varphi = 0, \ I = 0,$$

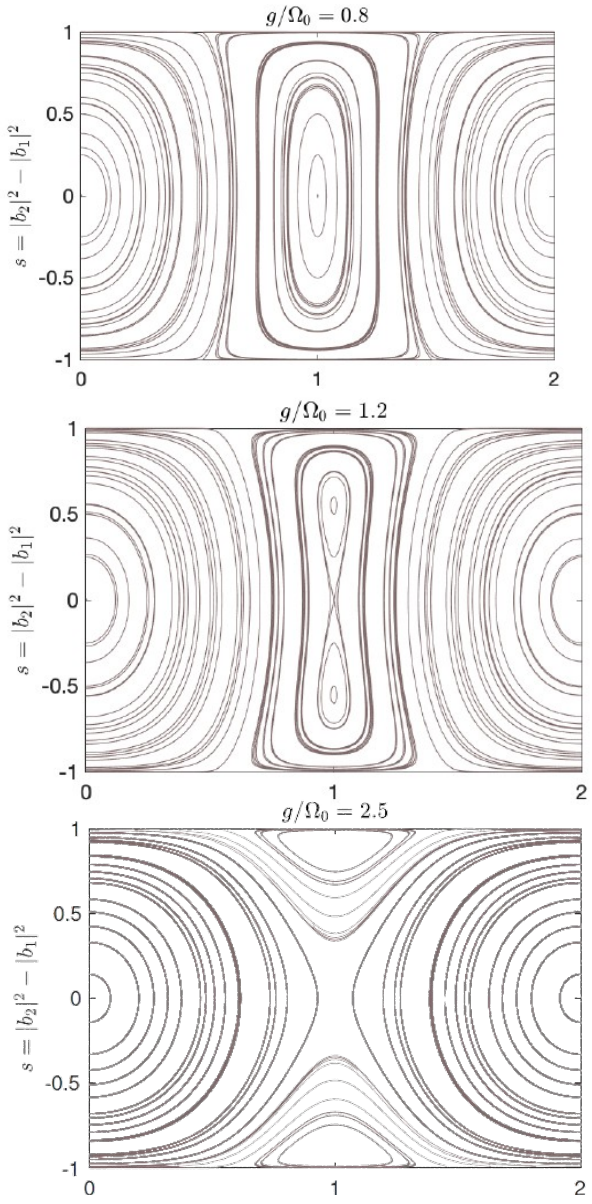
$$\varphi = \pi, \ I = I_0 \equiv 0 \text{ when } g \le \Omega,$$

$$\varphi = \pi, \ I = I_0 \equiv 0 \text{ or } I = I_\pm \equiv \pm\sqrt{1 - (\Omega/g)^2}$$

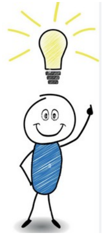
a phase transition happens at: $g = \Omega$

Adiabatic breakdown threshold

- Adiabaticity break down (not fast)
- No efficient state transfer



Nonlinear systems



Way out: **2K pulse and nonlinear Rabi oscillations**

- Parametrize the solution on a standard Bloch sphere:

$$\begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} = \begin{bmatrix} e^{i\varphi/2}\cos(\theta/2) \\ e^{-i\varphi/2}\sin(\theta/2) \end{bmatrix} e^{-i\gamma/2}, \quad \theta \in [0,\pi],\ \varphi \in [-\pi,\pi[,\ \gamma \in [-\pi,\pi[$$

↓ nonlinear Schordinger equation

$$\dot\theta = \Omega \sin\varphi,$$
$$\dot\varphi = \Omega \cot\theta \cos\varphi - \Delta + g\cos\theta,$$
$$\dot\gamma \sin\theta = \Omega\cos\varphi.$$

↓ $\Omega = \Omega_0 = \text{const.}$ $\quad \Delta = 0$

$$\dot\varphi = (\dot\gamma + g)\cos\theta,$$
$$\dot\theta^2 + \dot\gamma^2 \sin^2\theta = \Omega_0^2.$$

$\vartheta = -2\theta$ → $\vartheta = 2\theta$

$$\dot\vartheta^2 = 4\omega_0^2[1/m - \sin^2(\vartheta/2)]$$ rigid pendulum

$$\sin(\vartheta/2) = \sqrt{1/m}\,\mathrm{sn}\,(\omega_0 t; 1/m) = \mathrm{sn}\,(\Omega_0 t; m).$$

↑

$$P_{1\to2} = \frac{1}{2}\big(1 - \mathrm{cn}\,(\Omega_0 t, m)\big) = \mathrm{sn}^2\Big(\frac{\Omega_0 t}{2}, m\Big)\frac{1 - m\,\mathrm{sn}^2\big(\frac{\Omega_0 t}{2}, m\big)}{1 - m\,\mathrm{sn}^4\big(\frac{\Omega_0 t}{2}, m\big)}.$$

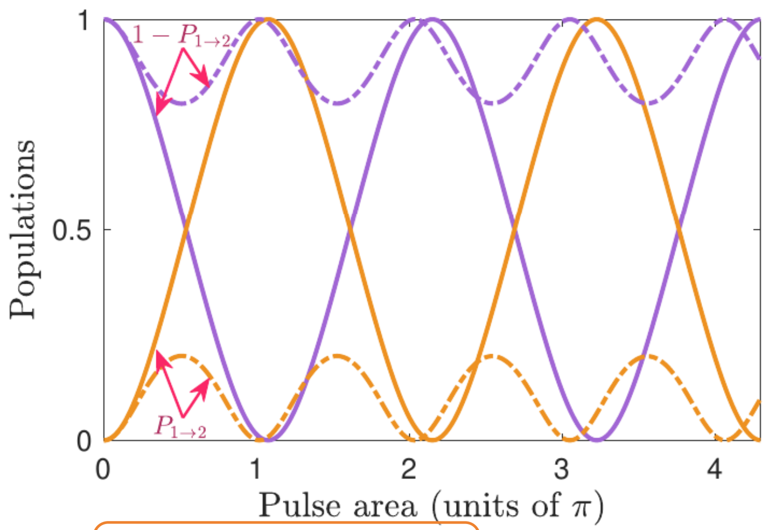
↓

$$\ddot\gamma + \dot\theta(2\dot\gamma + g)\,\mathrm{cotan}\,\theta = 0, \quad \rightarrow \quad \gamma(t) = -\frac{g}{2}t + \frac{\pi}{2}.$$

$$F(\theta|m) := \int_0^\theta \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} = \Omega_0 t, \quad m = [g/(2\Omega_0)]^2,$$

- chracterizes the nonlinearity in the system in terms of the oscillation period

Not robust!



K-threshold

m>=1 $\quad |g/\Omega_0| \geq 2,$

oscillations partial population

m<1 $\quad |g/\Omega_0| < 2,\ \Omega_0 t_{2K} = 2K(m)$

oscillations complete population

Nonlinear systems

Robust two-stage optimal inverse engineering:

- dynamically compensation of nonlinearity

linear two-level model → paramerization → dynamical equations

$$\theta(t) = \theta_\ell(t), \ \varphi(t) = \varphi_\ell(t)$$

$$H_\ell = \frac{\hbar}{2}\begin{bmatrix} \Delta_\ell & \Omega_\ell \\ \Omega_\ell & -\Delta_\ell \end{bmatrix}$$

$$\begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} = \begin{bmatrix} e^{i\varphi_\ell/2}\cos(\theta_\ell/2) \\ e^{-i\varphi_\ell/2}\sin(\theta_\ell/2) \end{bmatrix} e^{-i\gamma_\ell/2}$$

$$\dot{\theta}_\ell = \Omega_\ell \sin\varphi_\ell,$$
$$\dot{\varphi}_\ell = \Omega_\ell \cot\theta_\ell \cos\varphi_\ell - \Delta_\ell,$$
$$\dot{\gamma}_\ell \sin\theta_\ell = \Omega_\ell \cos\varphi_\ell.$$

nonlinear two-level model → paramerization → dynamical equations

$$\Omega = \Omega_\ell,$$
$$\Delta = \Delta_\ell + g\cos\theta_\ell = \Delta_\ell - g(|a_2|^2 - |a_1|^2).$$

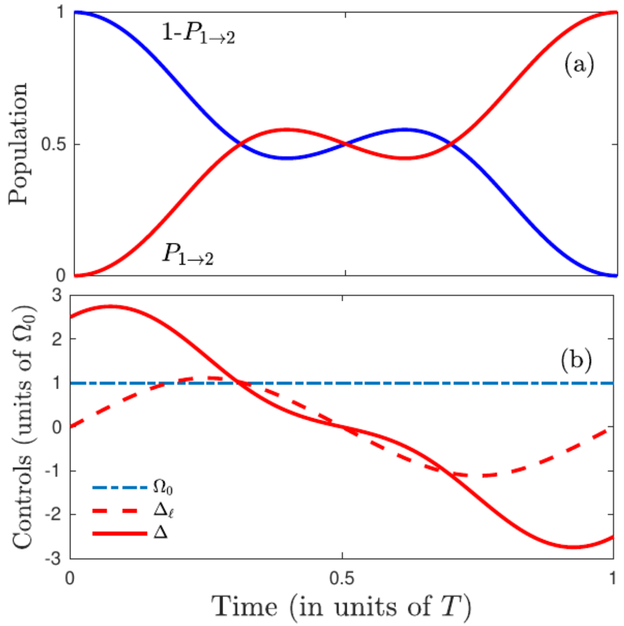
$$\begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix} = \begin{bmatrix} e^{i\varphi/2}\cos(\theta/2) \\ e^{-i\varphi/2}\sin(\theta/2) \end{bmatrix} e^{-i\gamma/2}$$

$$\dot{\theta} = \Omega \sin\varphi,$$
$$\dot{\varphi} = \Omega \cot\theta \cos\varphi - \Delta + g\cos\theta,$$
$$\dot{\gamma}\sin\theta = \Omega\cos\varphi.$$

Nonlinear RIO pulses: ← RIO pulses:

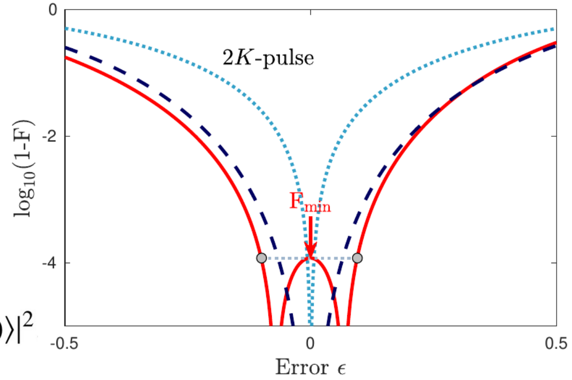
$$\Omega_\ell = \Omega_0 = 5.84/T,$$
$$\Delta_\ell(t) = -\Delta_0 \mathrm{cn}(4K(m)t/T + K(m), m),$$

$$g = 2.5\Omega_0$$



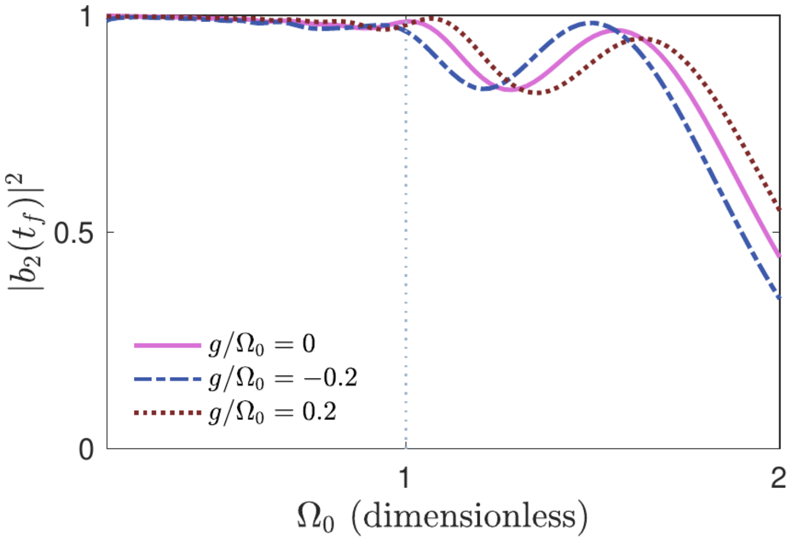
$$g = -0.2\Omega_0$$

$$F = |\langle 2|\phi(t = t_f)\rangle|^2$$



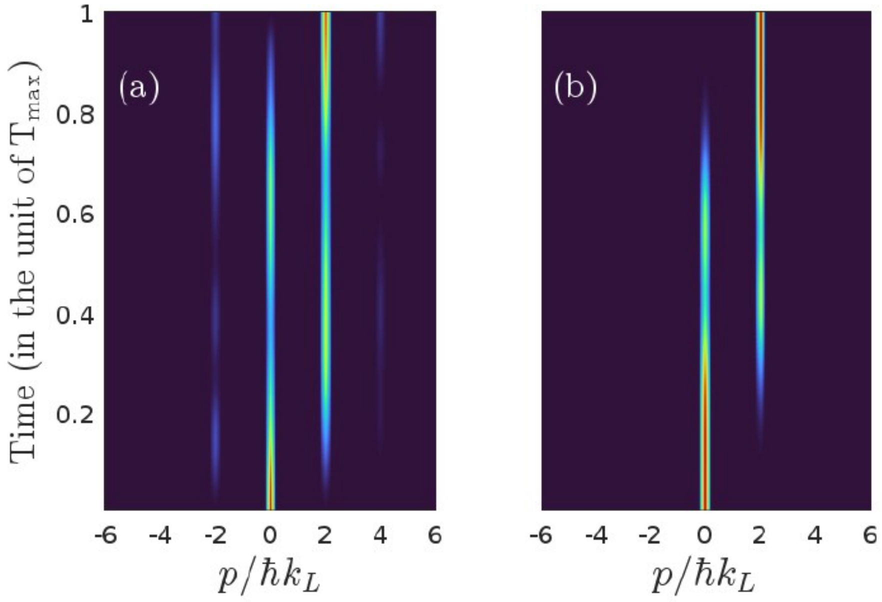
Nonlinear systems

Experimental realization:



$$T = 5.84 \times 2\hbar/E_R \approx 500\mu s$$

$$\Omega_0 = 1$$



- For deeper lattice, two modes does not hold
- Dealing with leakage problems

Outline

Conclusion

Present work:

- Robust time-optimal control for a single qubit system
- Robust control in the nonlinear two-level system
- Direct application to control motional states of BECs in the accelerated optical lattice

THANK YOU

merci

For your attention