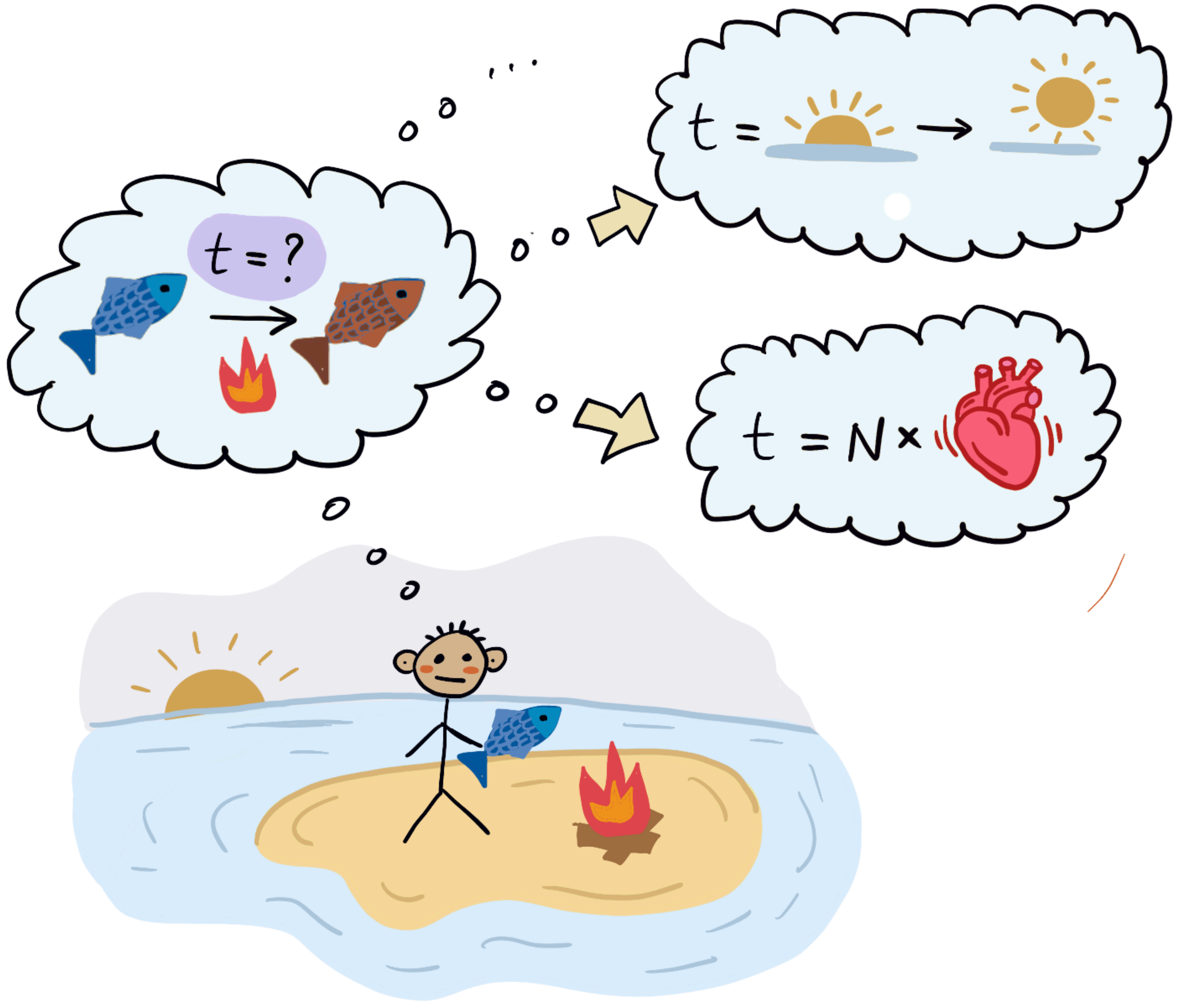
|T⏰CQS⟩

Thermodynamics of precision

From counting statistics to clocks



Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

ASPECTS
aspects-quantum.com

sfi Fondúireacht Eolaíochta Éireann
Dá bhfuil romhainn
Science Foundation Ireland
For what's next

UK Research and Innovation

THE ROYAL SOCIETY

Co-funded by the European Union

Mark T. Mitchison

Quantum Carnot Workshop

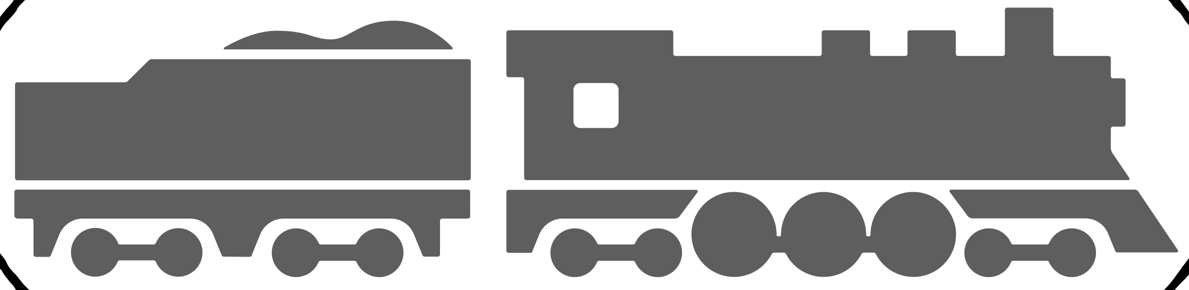
26/11/2024

Thermodynamics

Thermodynamics is a **practically motivated** physical theory:

What can we do, and **how well** can we do it, with available resources? 🔥 🔥 🔥 🔥

How efficiently can a steam engine convert fire into motion?



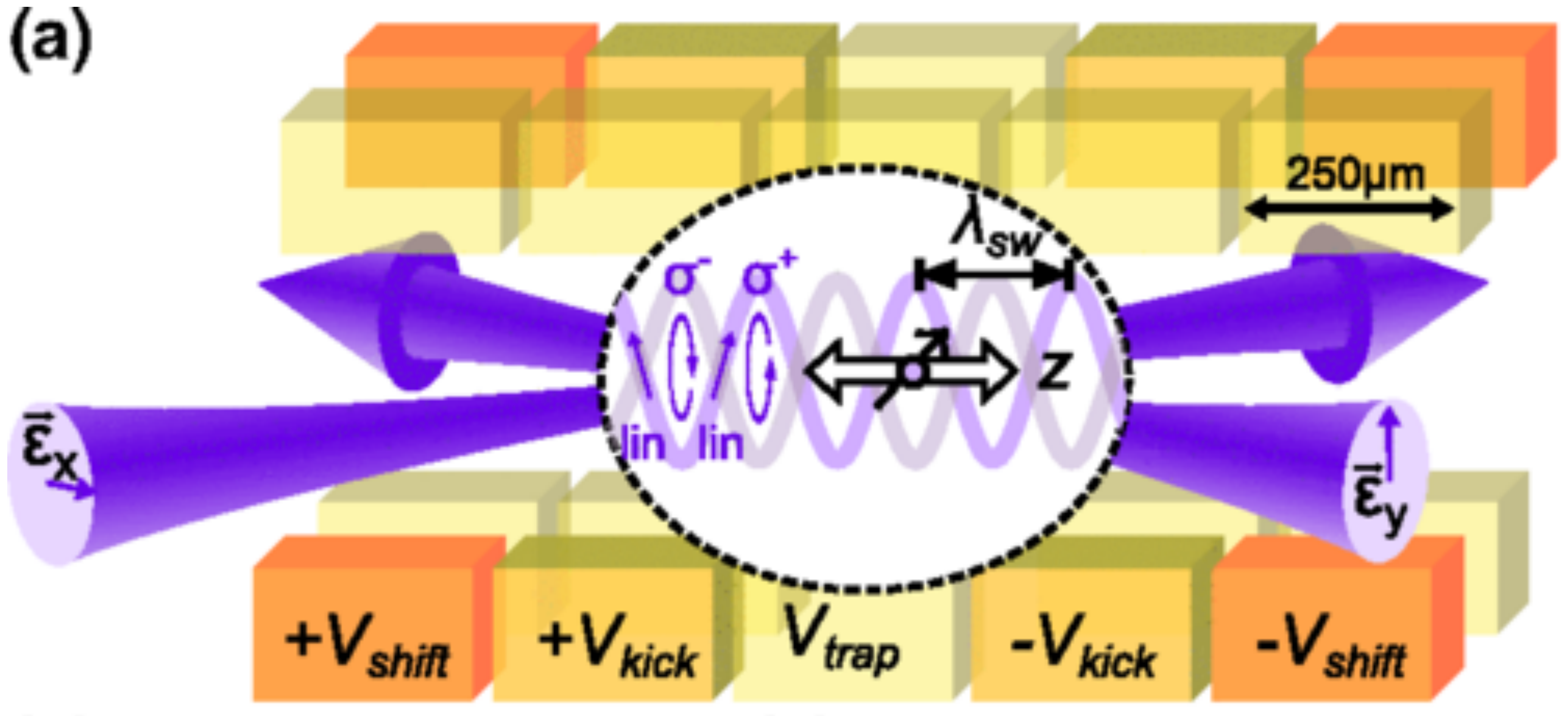
$$\eta \leq 1 - \frac{T_c}{T_h} \Leftrightarrow \boxed{\Delta S \geq 0}$$

- chemical reactions
- phase transitions
- black holes

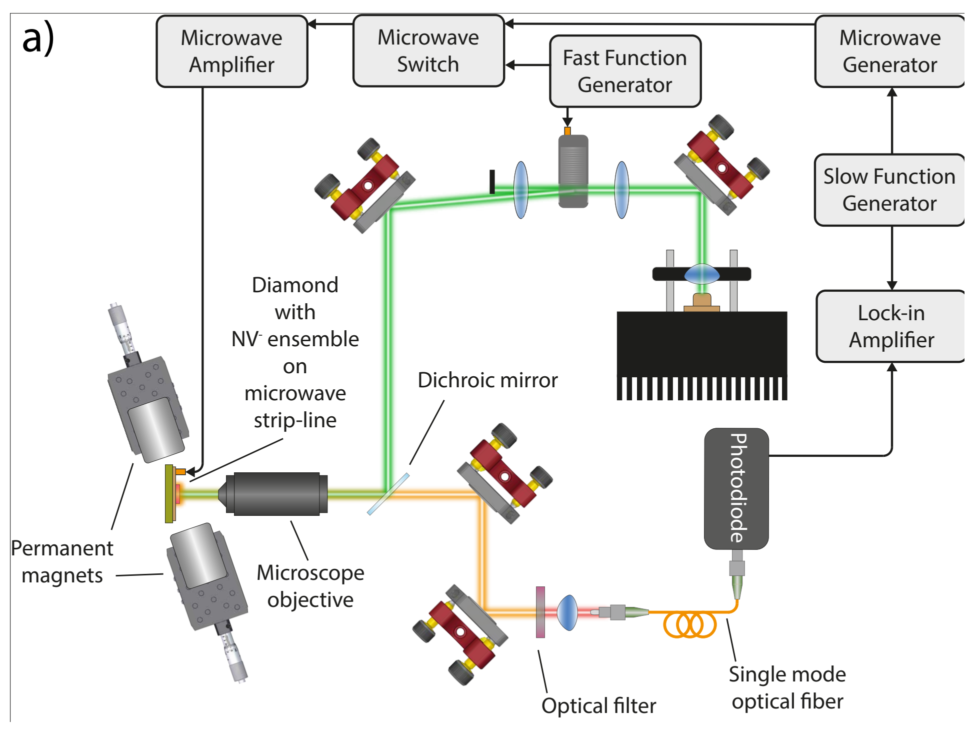
S. Carnot, *Reflexions sur la Puissance Motrice du Feu,* (1824)

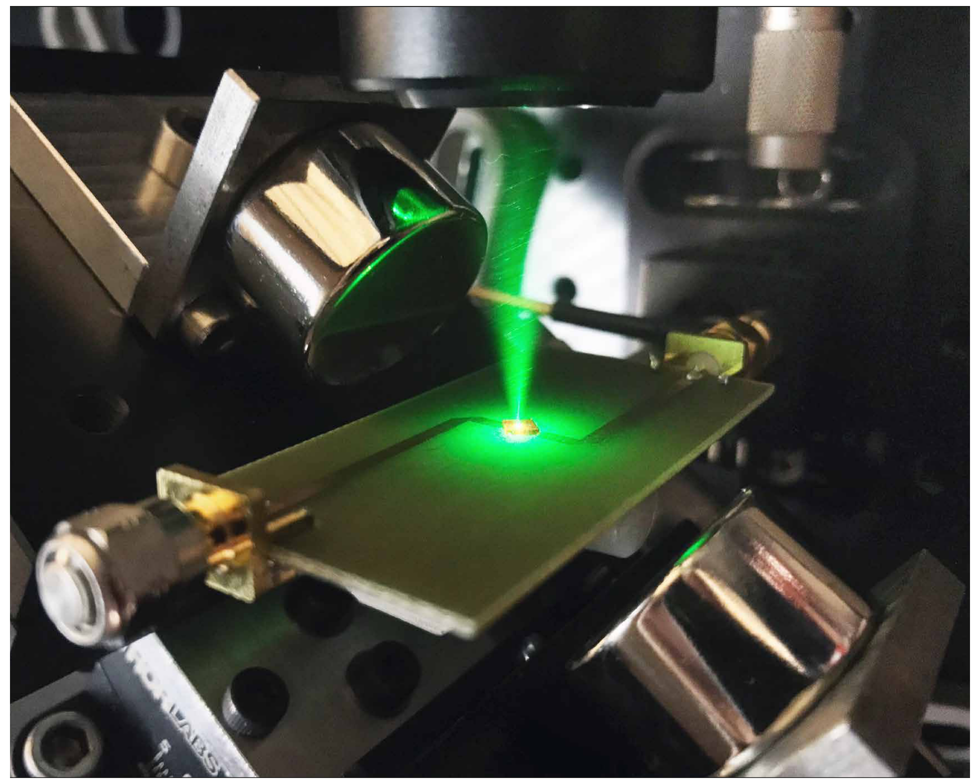
Quantum thermodynamics

What can we do, and **how well** can we do it, with **quantum** resources?

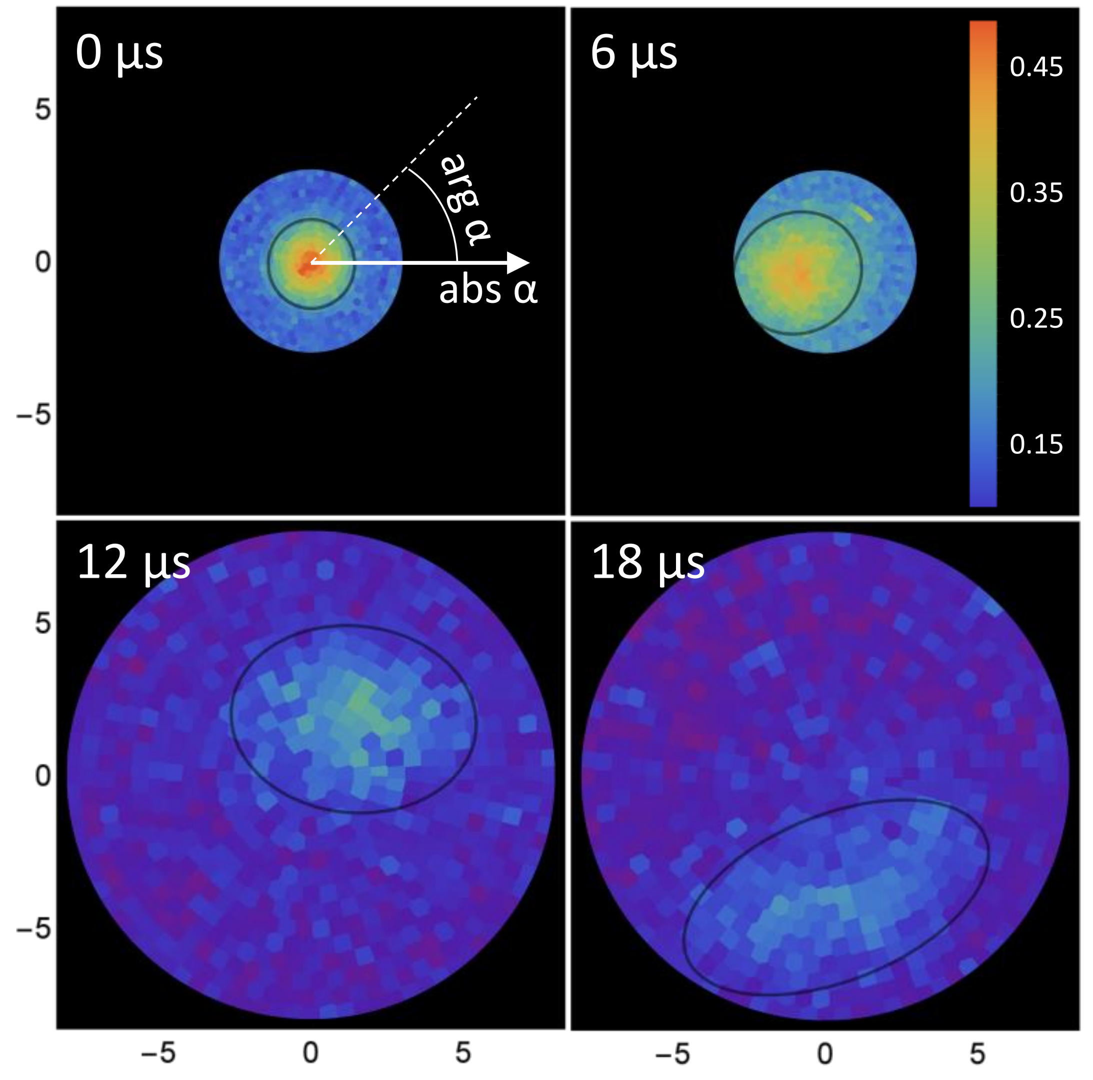


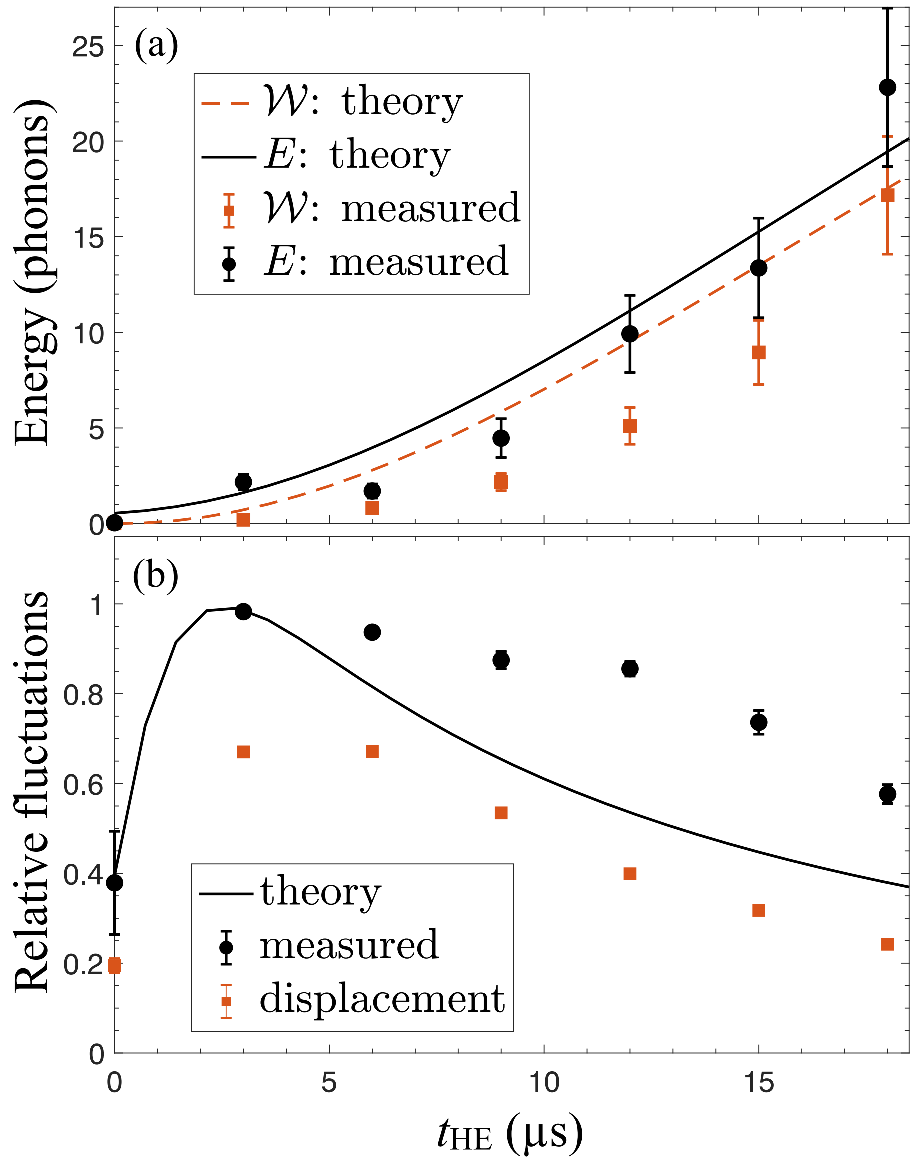
Schmiegelow et al., Phys. Rev. Lett. **116**, 033002 (2016)



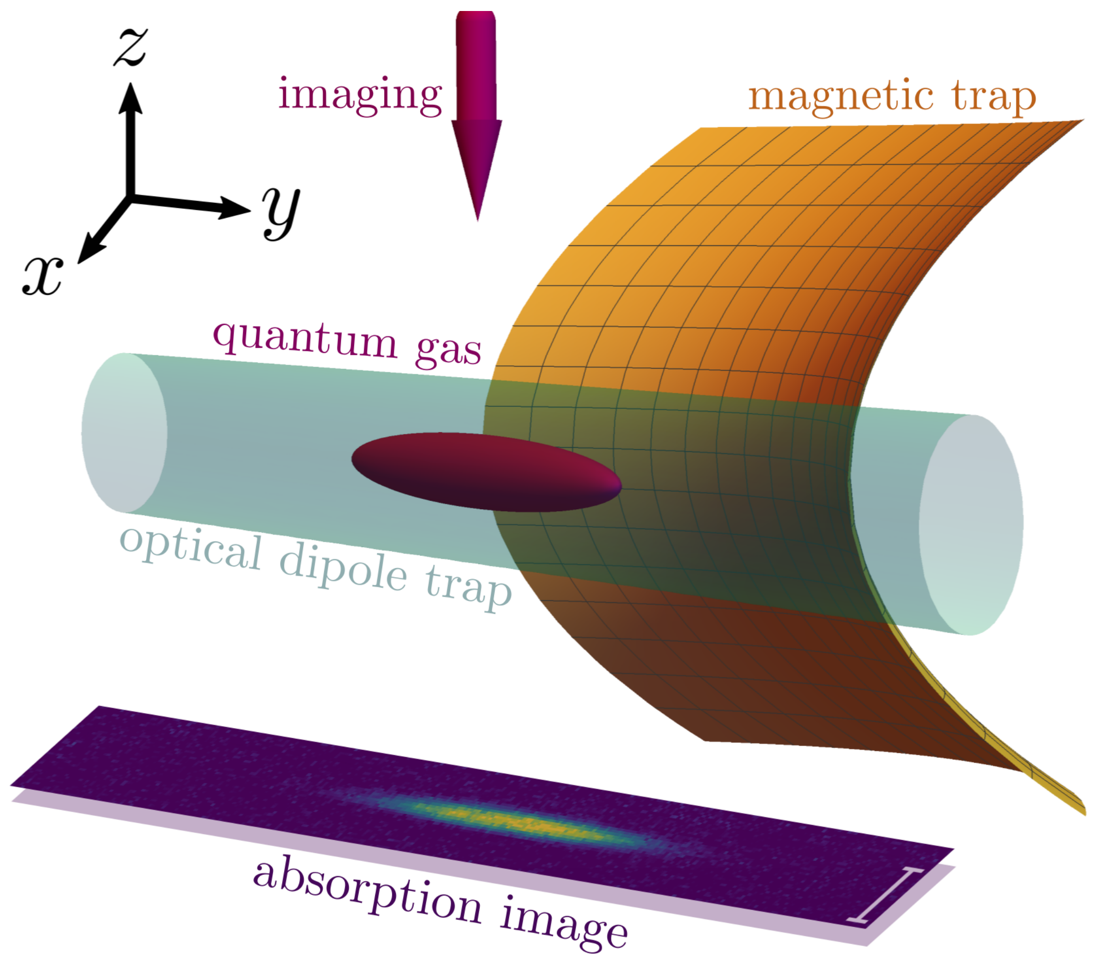


Klatzow, Phys. Rev. Lett. **122**, 110601 (2019)

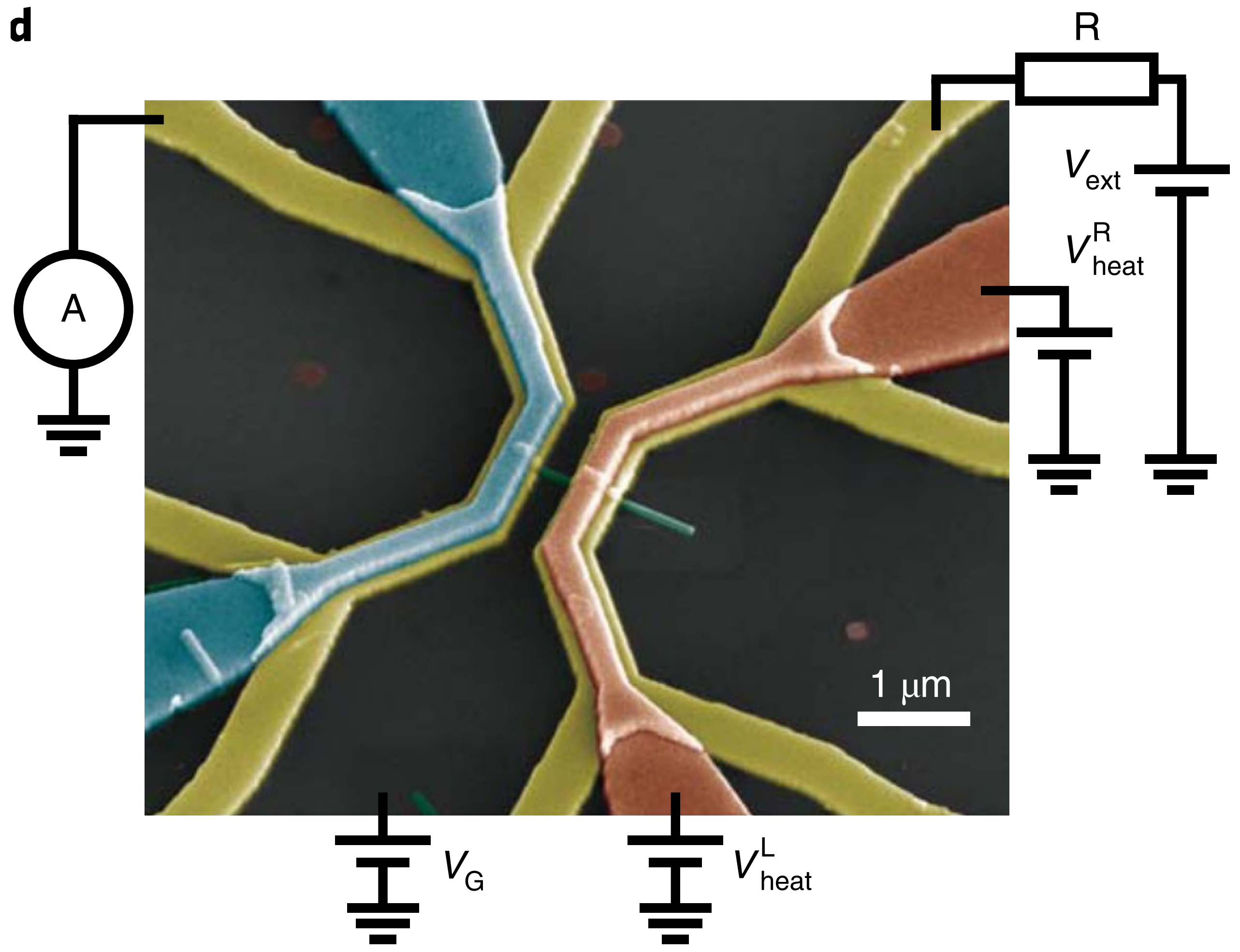




von Lindenfels et al., Phys. Rev. Lett. **123**, 080602 (2019)



Koch et al., Nature **621**, 723 (2023)



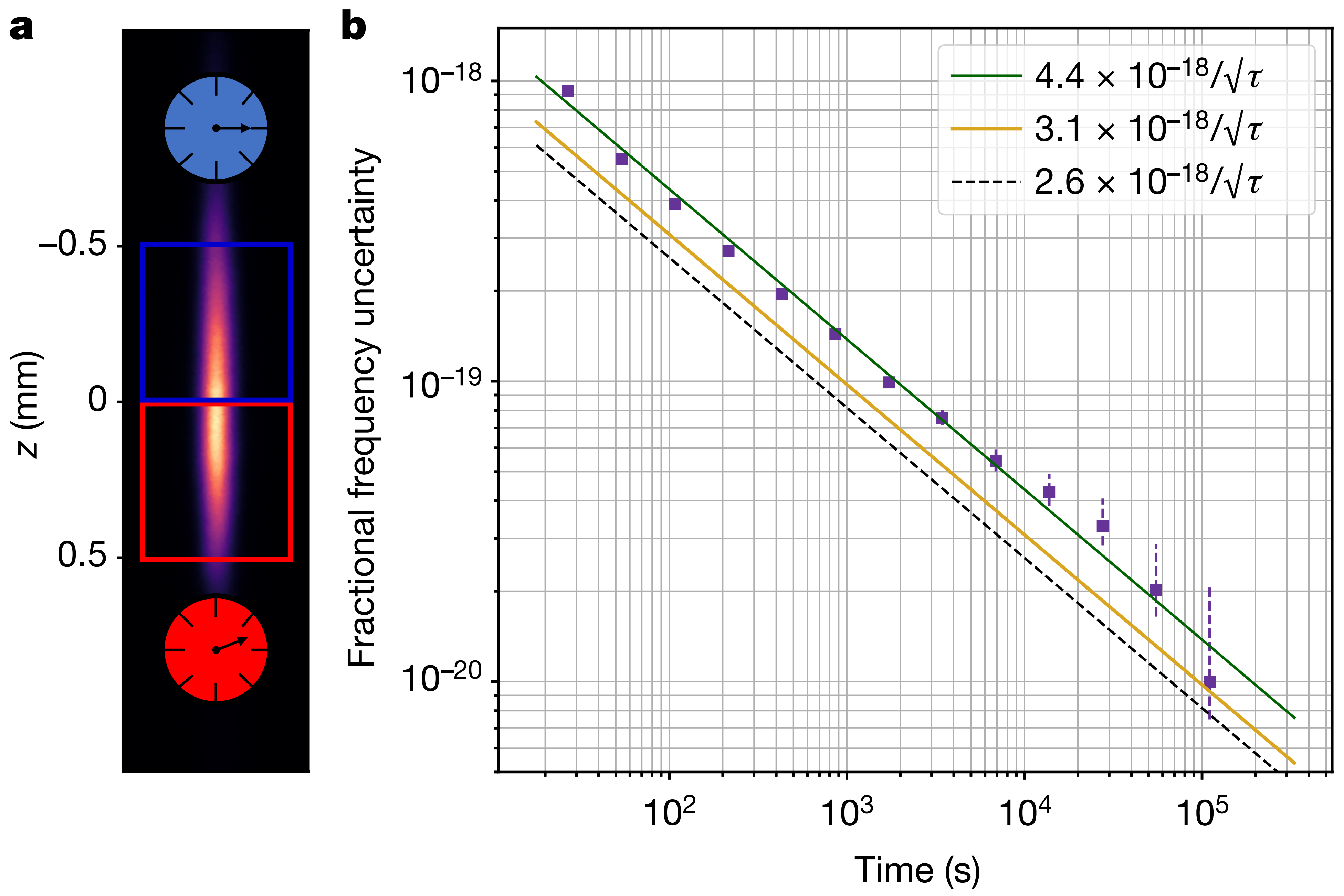
Josefsson et al.,
Nature Nanotech. **13**, 920 (2019)

Beyond heat engines

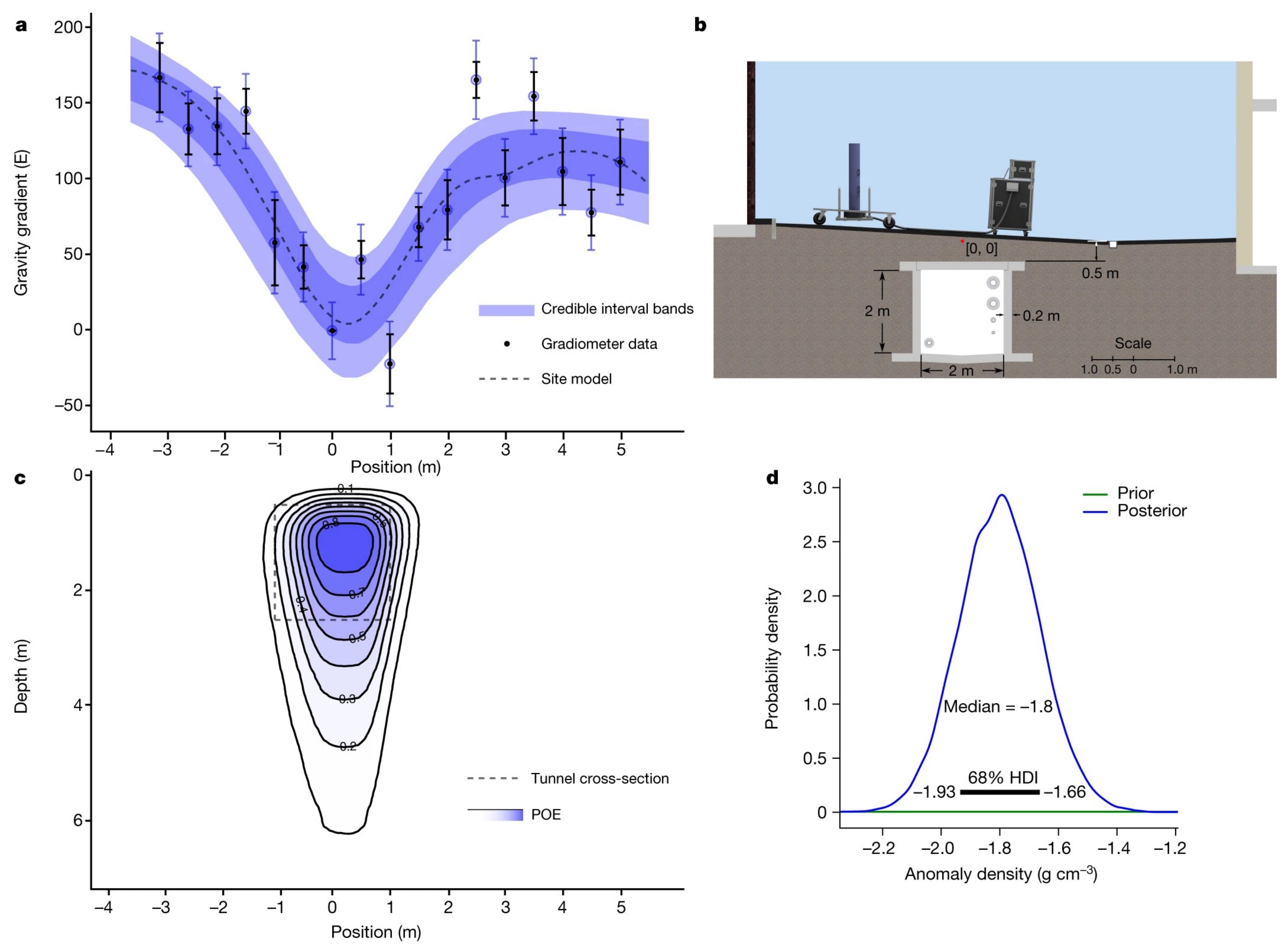
Does thermodynamics limit the efficiency of…

- entanglement/coherence generation
- quantum information processing
- **precision measurement**

…and what do we fundamentally learn if so?



T. Bothwell et al., *Resolving the gravitational redshift across a millimetre-scale atomic sample,* Nature **602**, 420 (2022)



B. Stray et al., *Quantum sensing for gravity cartography,* Nature **602,** 590 (2022)

Thermodynamics of precision

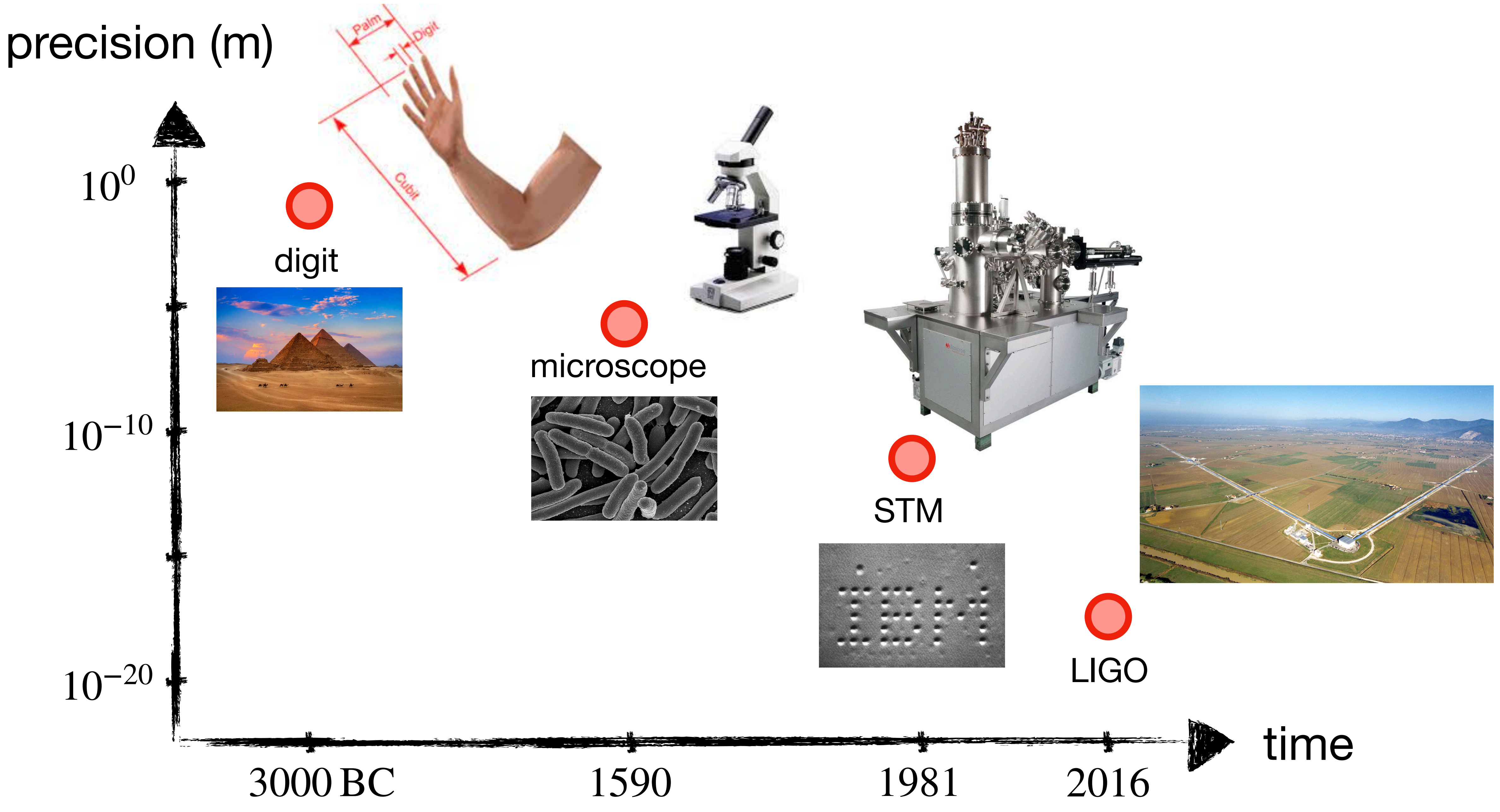
What are the physical limits of precision?

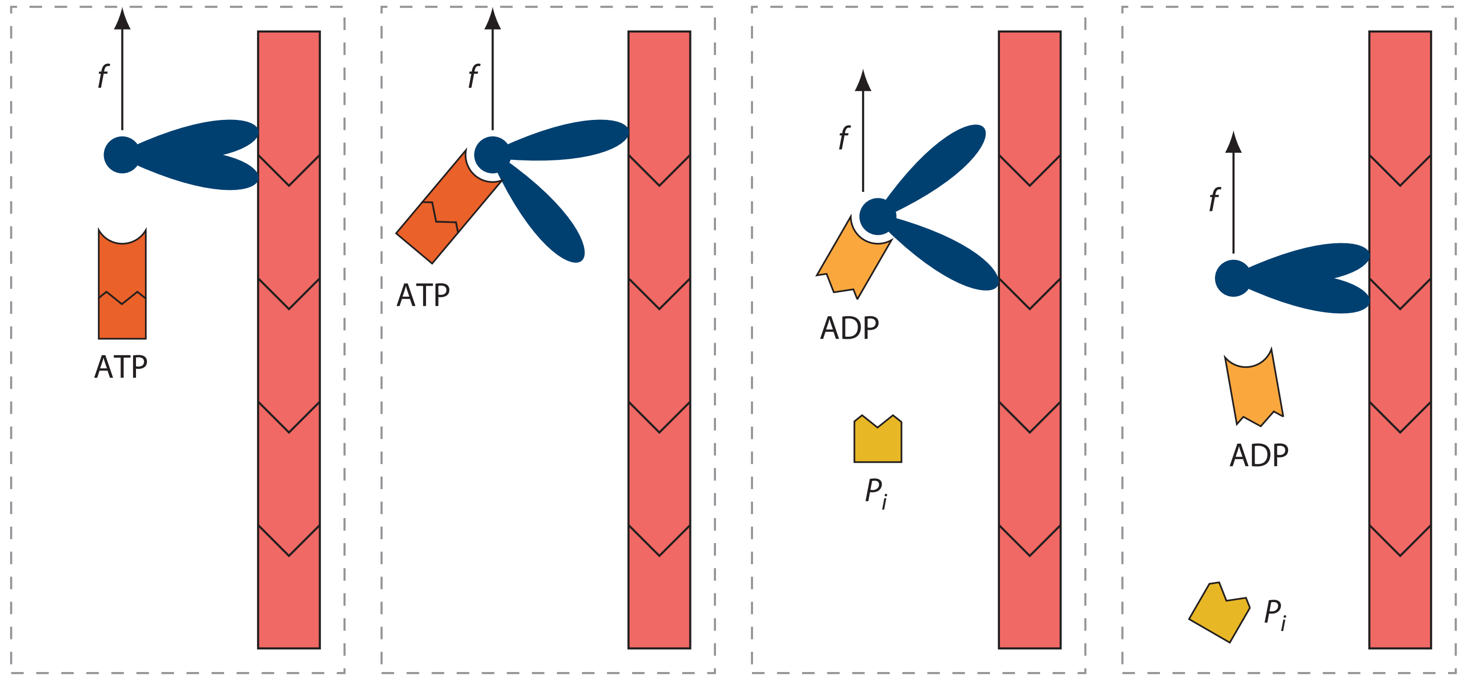
Quantum uncertainty relations: $\Delta q \cdot \Delta p \geq \dfrac{\hbar}{2}$

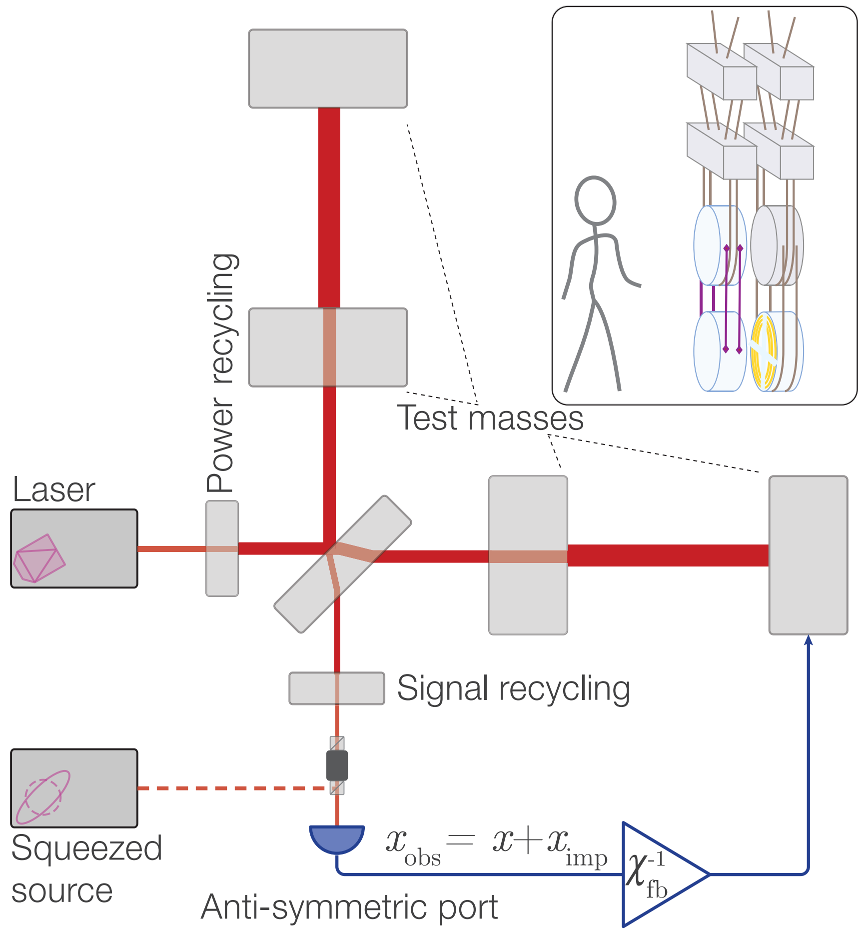
Thermodynamic uncertainty relations (TURs):

$$\boxed{\underbrace{\frac{\mathrm{Var}[\Theta]}{\mathrm{E}[\Theta]^2}}_{\text{imprecision}} \cdot \underbrace{\Sigma}_{\text{dissipation}} \geq 2k_B \quad \longrightarrow \quad \mathrm{SNR} \leq \frac{\dot{\Sigma}}{2k_B}}$$

Seifert, Annu. Rev. Condens. Matter Phys. **10** 171 (2019)
Horowitz & Gingrich, Nature Phys. **16**, 15 (2020)







Quantum thermodynamics → "coherence" allows higher SNR for given $\dot{\Sigma}$

Ptaszynski,
Phys. Rev. B **98**, 085425 (2018)

Agarwalla & Segal,
Phys. Rev. B **98**, 155438 (2018)

Brandner et al.,
Phys. Rev. Lett. **120**, 090601 (2018)

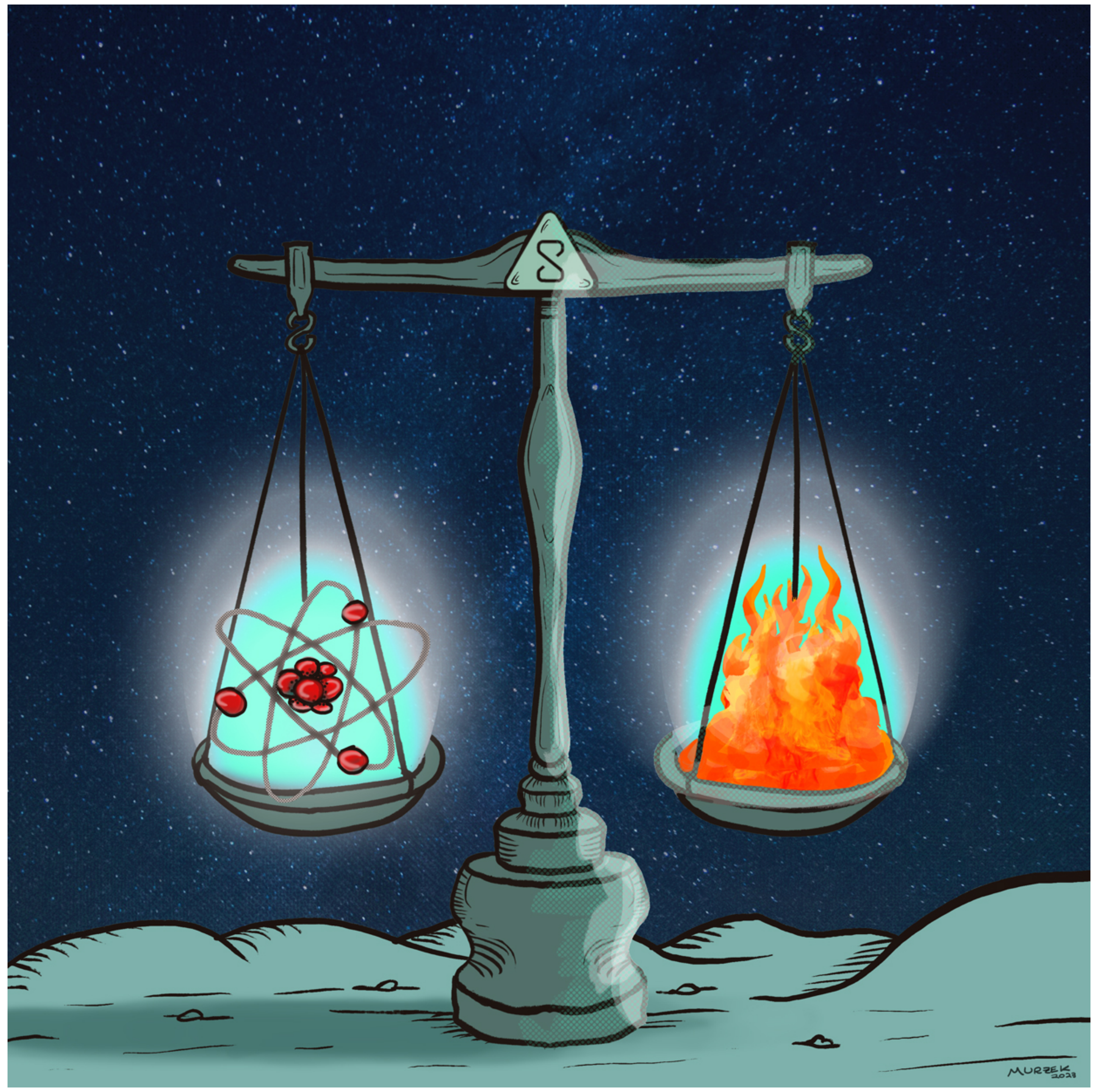
LIGO, Science **372**, 1333 (2021)

ASPECTS

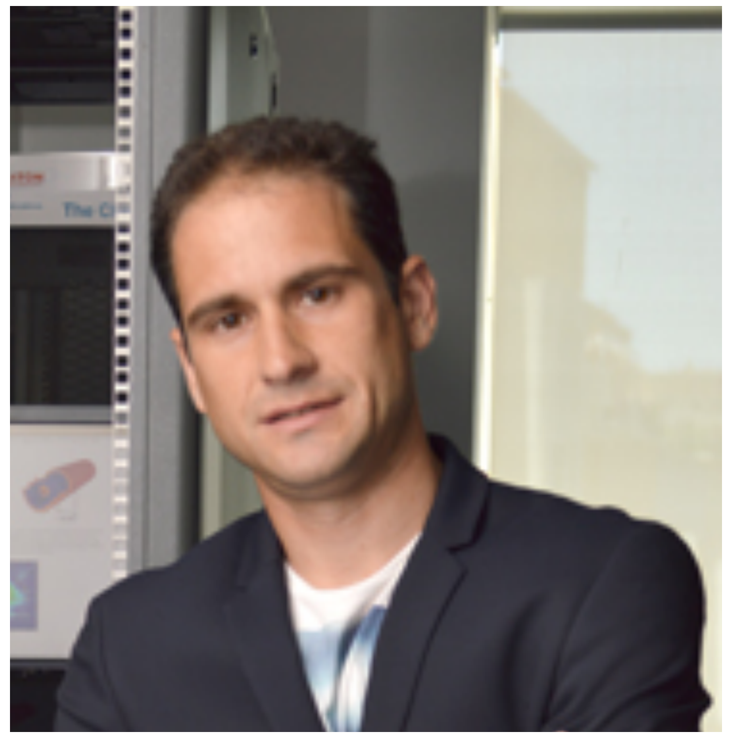
Huber
(TU Wien)

Gasparinetti
(Chalmers)

me



Ares
(Oxford)



Prior
(Murcia)

Apollaro
(Malta)

QUANTUM FLAGSHIP

UK Research and Innovation

Co-funded by the European Union

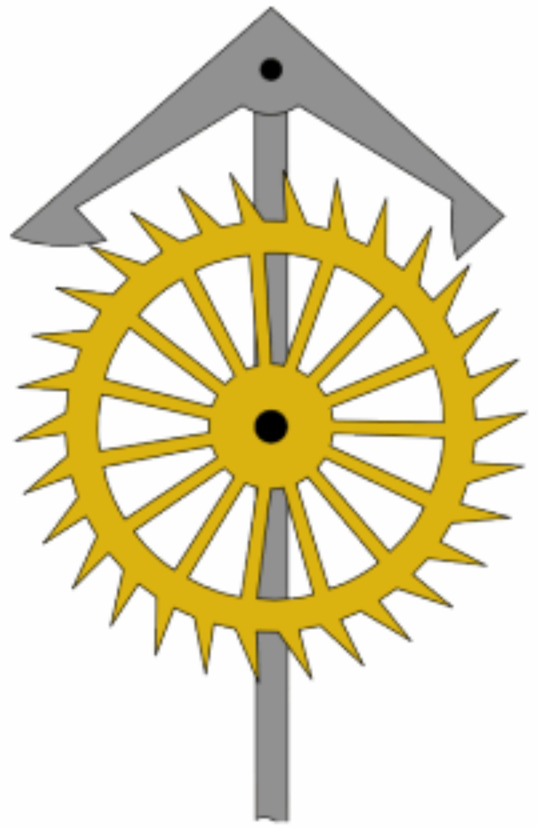
Thermodynamics of clocks

|T◉CQS⟩

A clock is a machine that converts energy into a sequence of ticks

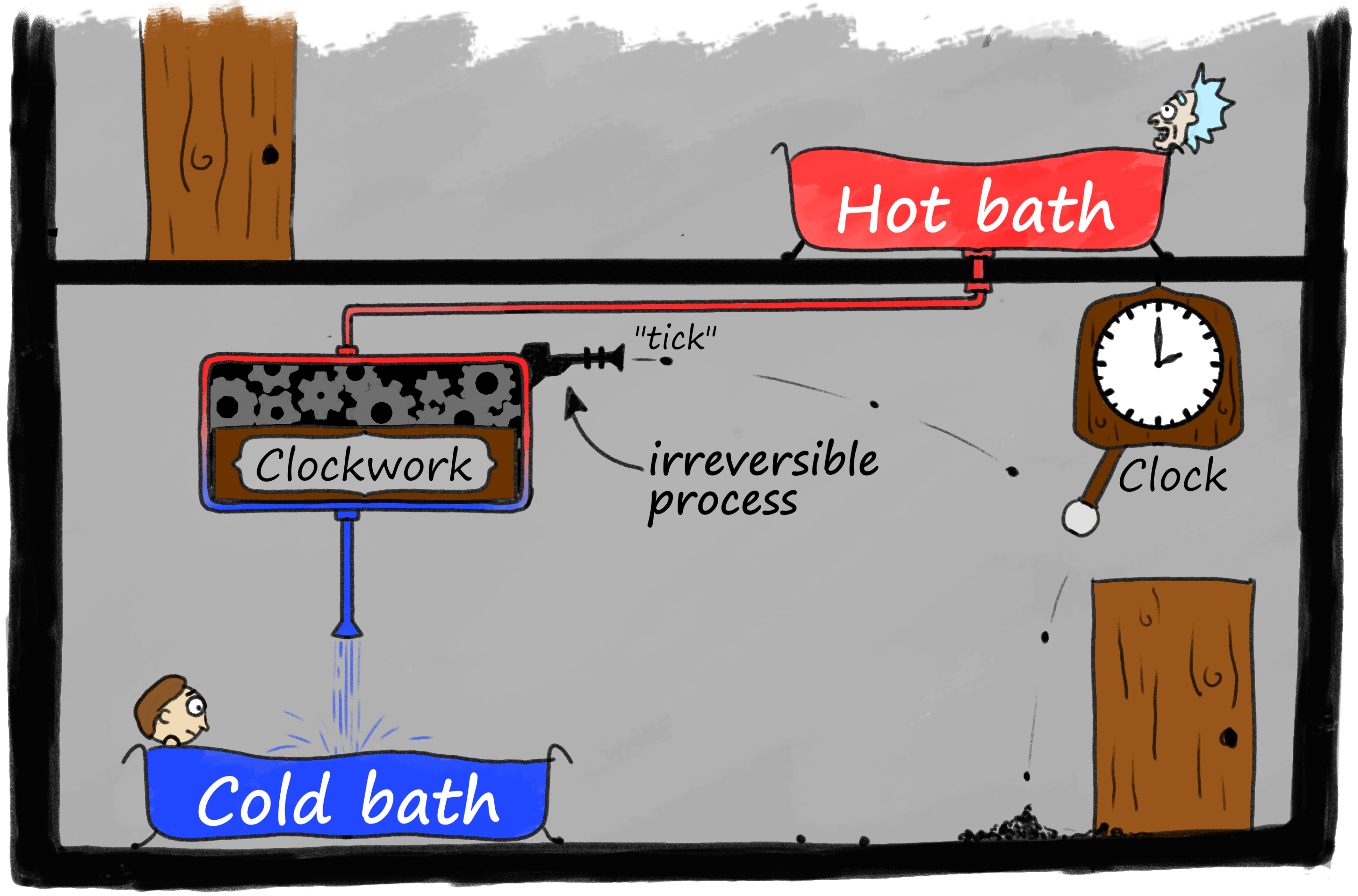
What are the minimal resources? How efficient can a clock be?

What happens as we scale clocks down to the quantum domain?



Need an **autonomous** framework to ensure fair bookkeeping:

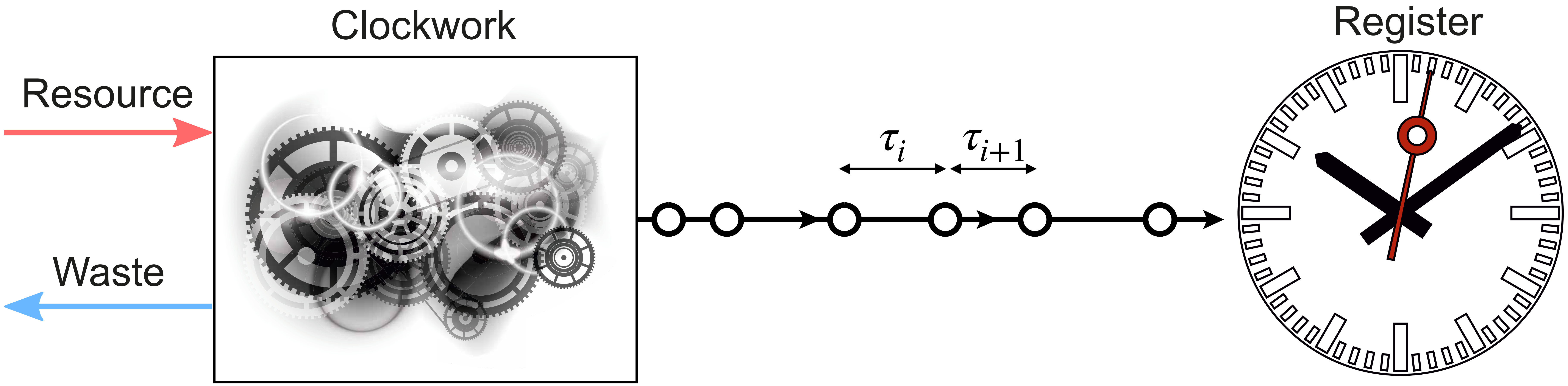
- ticks are generated continuously and recorded spontaneously in a register
- resources powering the clockwork are time independent



Erker, MTM, Silva, Woods, Brunner & Huber, Phys. Rev. X **7**, 031022 (2017)
Milburn, Contemp. Phys. **61**, 69 (2019)
Pearson, et al., Erker, Huber & Ares, Phys. Rev. X **11**, 021029 (2019)
Schwarzhans, Lock, Erker, Friis & Huber, Phys. Rev. X **11**, 011046 (2021)

Quantifying performance

|T⏰CQS⟩



mean waiting time: $\mu_i = \langle \tau_i \rangle$ **variance:** $\sigma_i^2 = \langle (\tau_i - \mu_i)^2 \rangle$

Assuming clockwork is a renewal process: all τ_i are i.i.d. random variables

Resolution $\nu = \mu^{-1}$ (average tick frequency)

Accuracy $\mathcal{N} = \dfrac{\mu^2}{\sigma^2}$ (number of ticks before the clock is off by one tick)

c.f. Allan variance $\sigma_y^2(T) \sim \dfrac{\mu}{NT}$ for large T

Erker, MTM, Silva, Woods, Brunner & Huber,
Phys. Rev. X **7**, 031022 (2017)

Clocks & counting statistics

Can also consider full counting statistics of the "tick current" $I(t) = \dfrac{\mathrm{d}N}{\mathrm{d}t}$

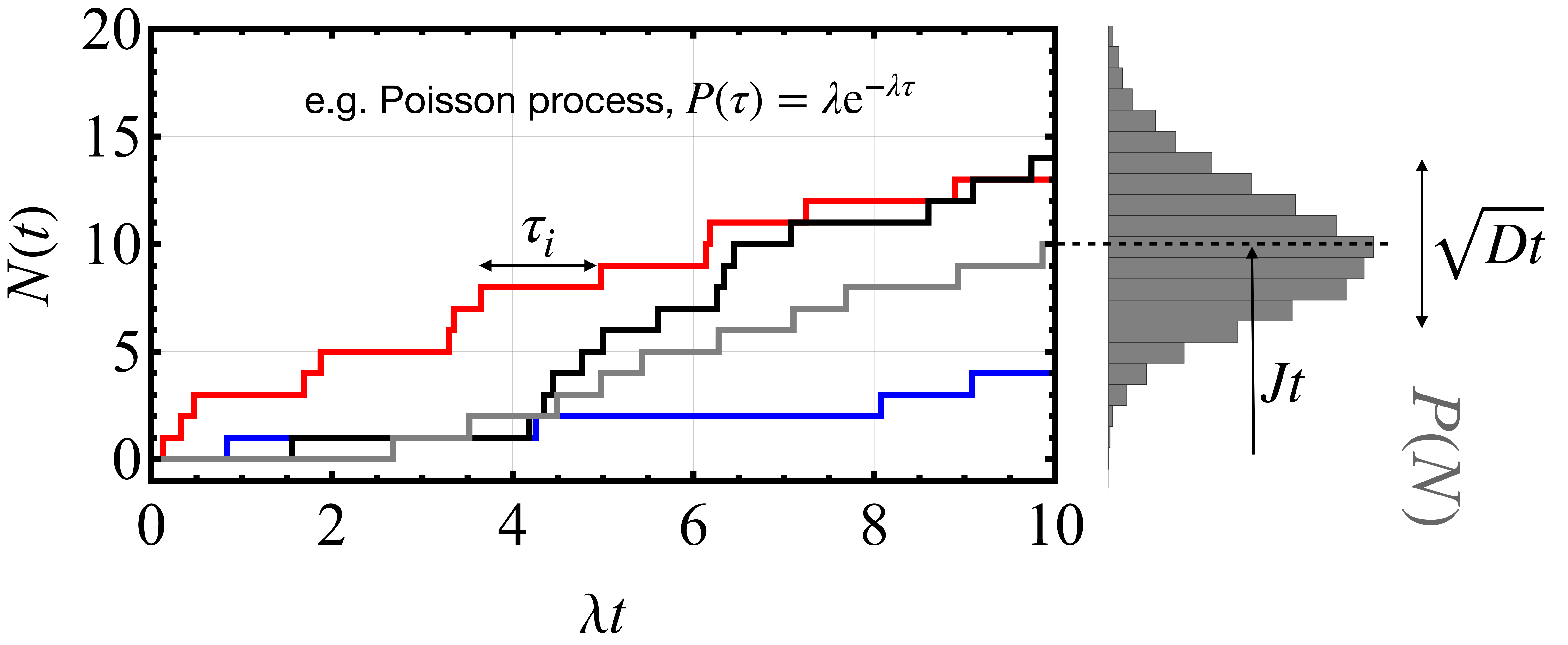
Landi, Kewming, MTM & Potts, PRX Quantum **5**, 020201 (2024)

Mean current:

$$J = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[N(t)]$$

Current noise/diffusion:

$$D = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}[N(t)]$$



Resolution $\nu = J$ (average tick frequency)

Accuracy $\mathcal{N} = \dfrac{J}{D}$ (number of ticks before the clock is off by one tick)

These definitions work for general (e.g. non-renewal) processes

Silva, Nurgalieva & Wilming, arXiv:2306.01829

Classical precision bounds

|T🕒CQS⟩

Thermodynamic uncertainty relation (TUR): accuracy is limited by dissipation

$$\frac{\mathrm{Var}[N]}{\mathrm{E}[N]^2} \cdot \Sigma \geq 2k_B \quad \xrightarrow[t\to\infty]{} \quad \frac{J^2}{D} \leq \frac{\dot{\Sigma}}{2k_B} \implies \boxed{\mathcal{N} \leq \frac{\Sigma_{\text{tick}}}{2k_B}}$$

$$\Sigma_{\text{tick}} = \mu \dot{\Sigma}$$

mean tick time

entropy production rate

Barato & Seifert, Phys. Rev. Lett. **114**, 158101 (2015)
Gingrich et al., Phys. Rev. Lett. **116**, 120601 (2016)

Erker et al., Phys. Rev. X **7**, 031022 (2017)
Pearson et al., Phys. Rev. X **11**, 021029 (2019)

Quantum dynamics allows exponential improvement $\mathcal{N} = \mathrm{e}^{\mathcal{O}(\Sigma_{\text{tick}})}$

Meier, Minoguchi, Sundelin, Apollaro, Erker, Gasparinetti & Huber, arXiv:2407.07948

Kinetic uncertainty relation (KUR): accuracy-resolution tradeoff

Garrahan, Phys. Rev. E **95**, 032134 (2017)
Terlizzi & Baiesi, J. Phys. A **52** 02LT03 (2019)

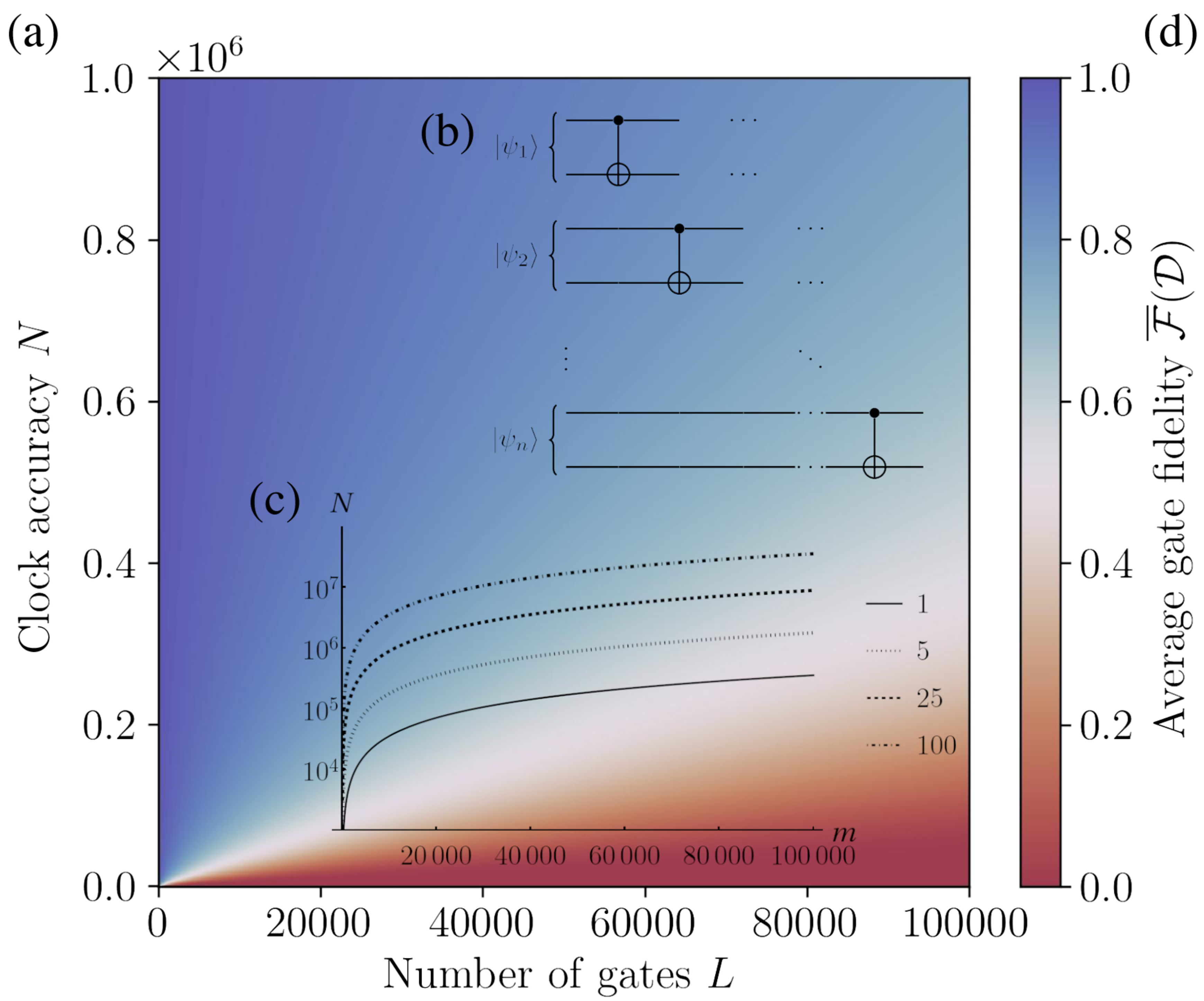
$$\frac{J^2}{D} \leq \mathscr{A} \implies \boxed{\mathcal{N}\nu \leq \mathscr{A}}$$

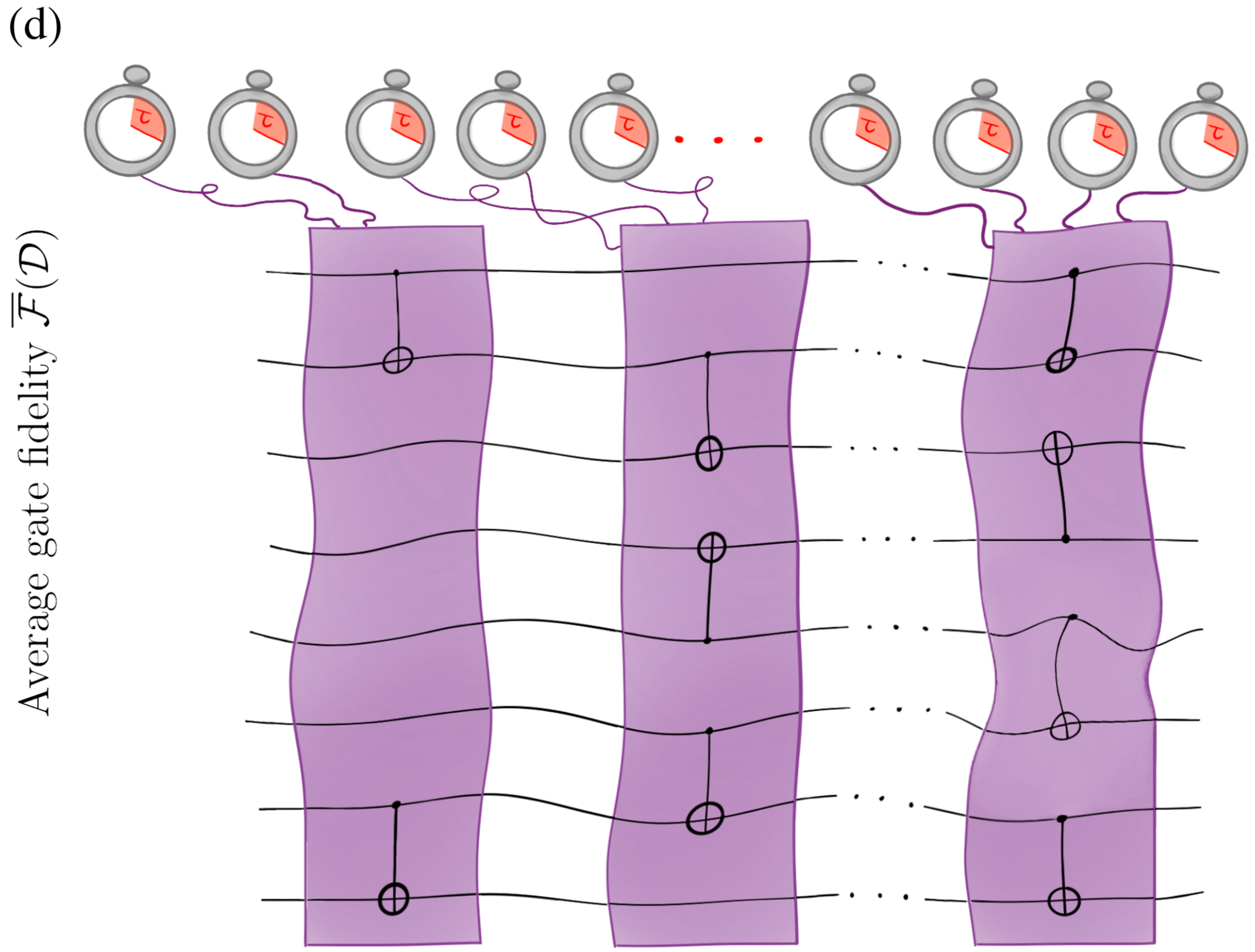
dynamical activity = (# of transitions)/time

Quantum dynamics allows quadratic improvement $\mathcal{N} \leq \left(\frac{A}{\nu}\right)^2$

Meier, Schwarzhans, Erker & Huber, Phys. Rev. Lett. **131**, 220201 (2023)

Implications for quantum control





$$U_t|\psi\rangle \rightarrow \int \mathrm{d}t\, p(t)\, U_t|\psi\rangle\langle\psi|\, U_t^\dagger$$

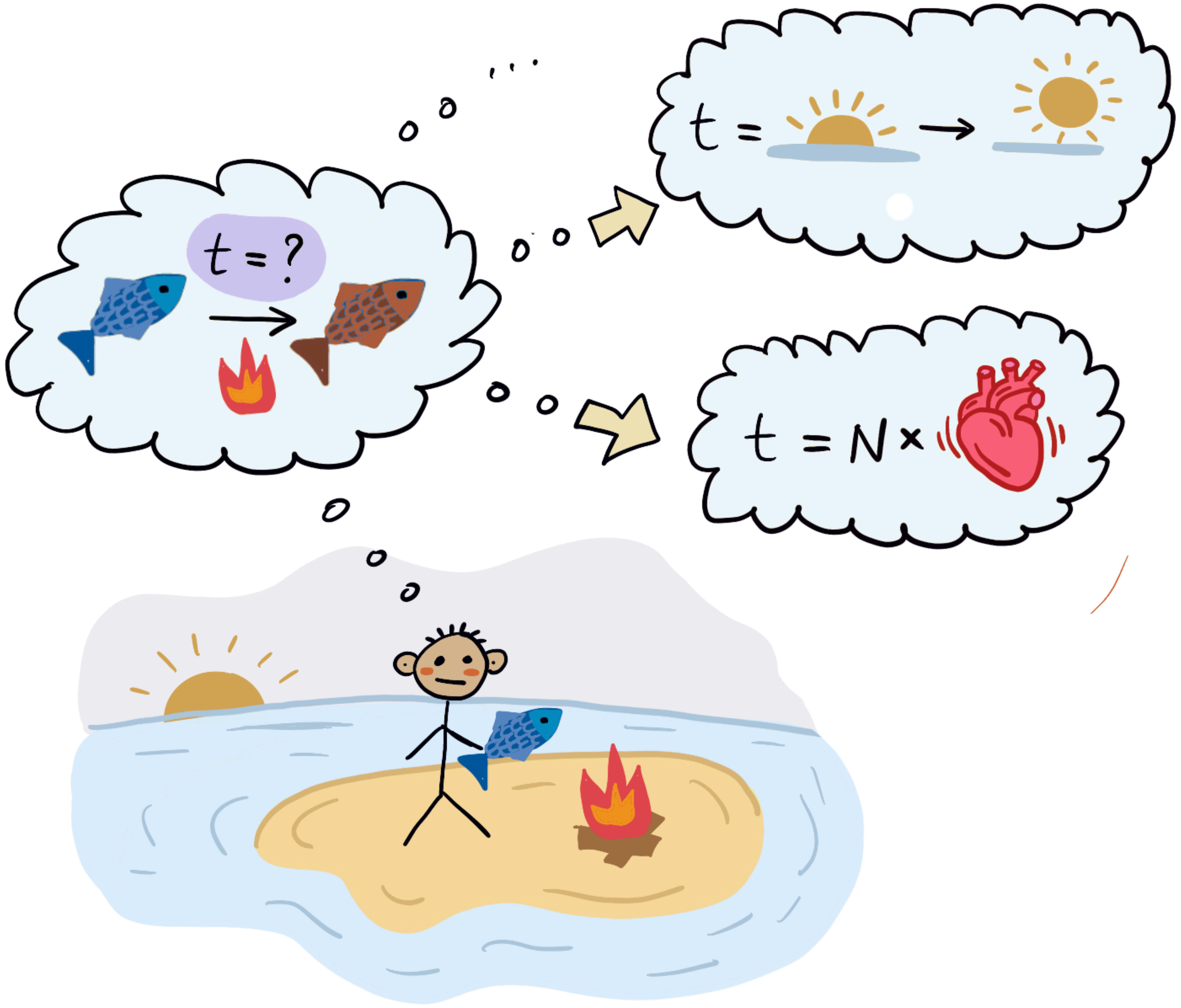
Generic circuit with L ill-timed CNOT gates has an average fidelity $\overline{\mathcal{F}} \leq \left(\frac{1}{2}\left[1 + e^{-\pi^2/\mathcal{N}}\right]\right)^{\frac{L}{2}}$

Xuereb, Meier, Erker, MTM & Huber, Phys. Rev. Lett. **131**, 160204 (2023)

See also:
Xuereb, Debarba, Huber & Erker, arXiv:2311.14561
Meier, Huber, Erker & Xuereb, arXiv:2402.00111

Stochastic time estimation

|T⌚CQS⟩

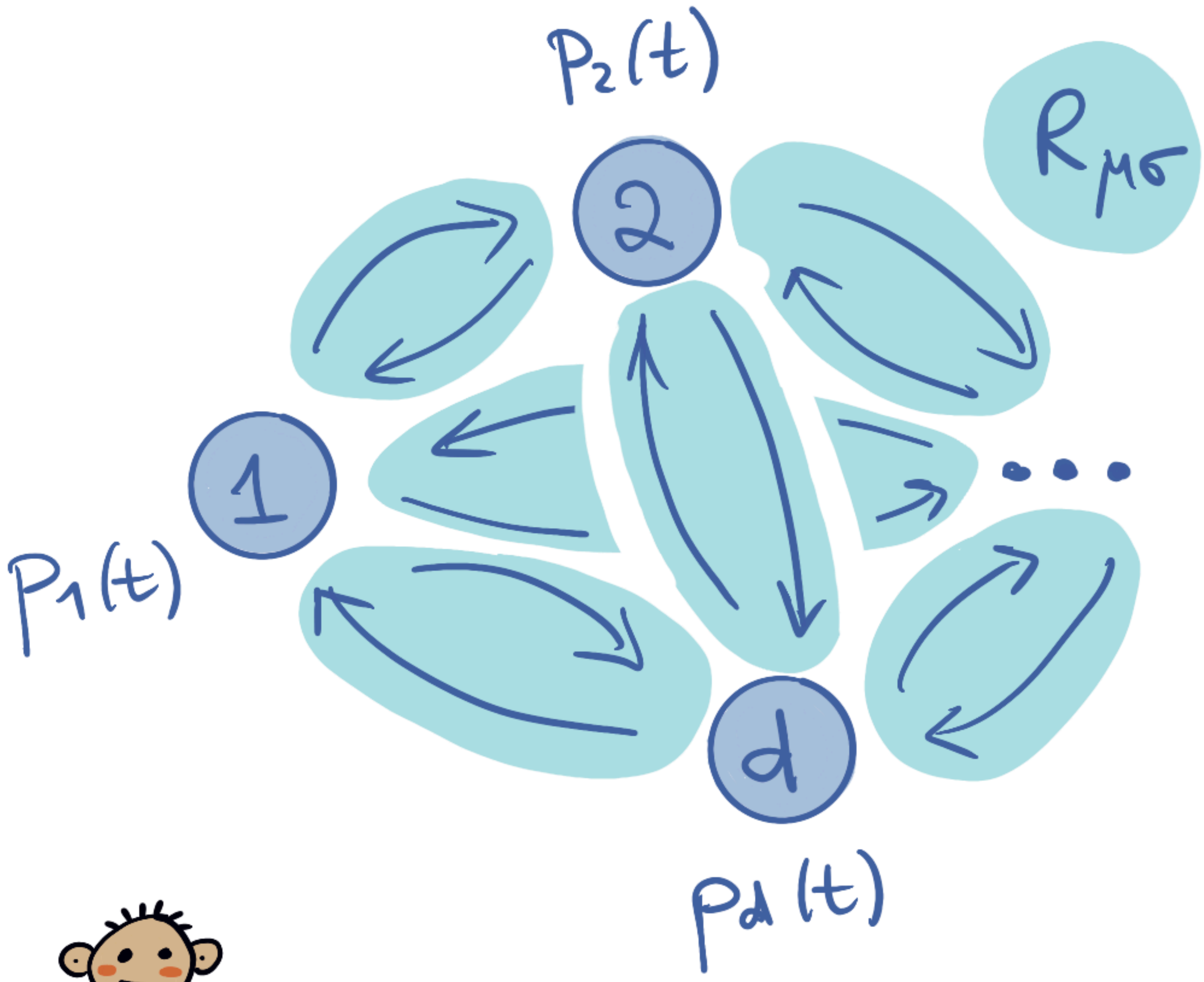


What if we can observe more than one type of “tick” event?

How to optimally (i.e. accurately & precisely) estimate time using these random events?

Markovian jump processes

|T⏰CQS⟩



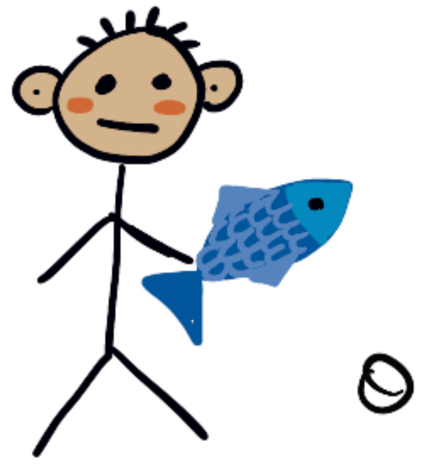
$$\partial_t p_\mu = \sum_{\sigma=1}^{d} L_{\mu\sigma} p_\sigma \qquad L_{\mu\sigma} = \begin{cases} R_{\mu\sigma} & (\mu \neq \sigma) \\ -\Gamma_\sigma & (\mu = \sigma) \end{cases} \qquad \Gamma_\sigma = \sum_{\mu=1}^{d} R_{\mu\sigma}$$

prob. to occupy state σ escape rate from σ jump rate $\sigma \to \mu$

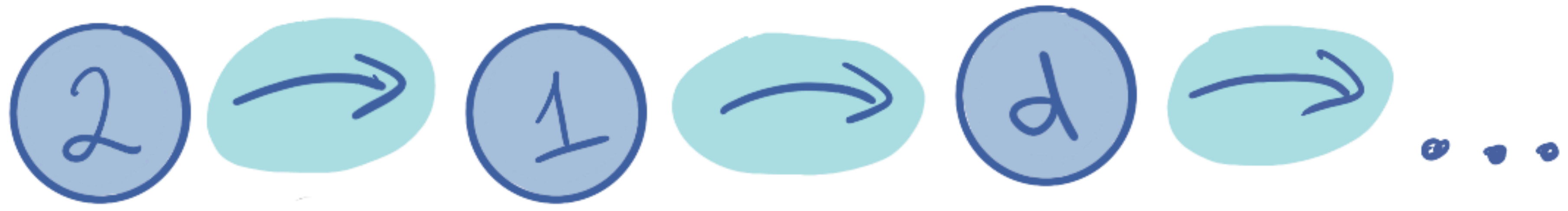
An observer lacking a clock can measure the ordered sequence

$$\boldsymbol{\sigma} = \sigma_0 \xrightarrow[t_1]{} \sigma_1 \xrightarrow[t_2]{} \cdots \xrightarrow[t_N]{} \sigma_N$$

$$\boxed{P(\boldsymbol{\sigma}\,|\,t) = \int_0^t dt_N \cdots \int_0^{t_2} dt_1\, P(\boldsymbol{\sigma}, \mathbf{t}\,|\,t)}$$



observed :



$$P(\boldsymbol{\sigma}, \mathbf{t}\,|\,t) = e^{-\Gamma_{\sigma_N}(t-t_N)} \prod_{j=0}^{N-1} \pi(\sigma_{j+1}\,|\,\sigma_j) W(t_{j+1} - t_j\,|\,\sigma_j) p^{\mathrm{ss}}_{\sigma_0}$$

$$\pi(\mu\,|\,\sigma) = \frac{R_{\mu\sigma}}{\Gamma_\sigma} \qquad W(\tau\,|\,\sigma) = \Gamma_\sigma e^{-\Gamma_\sigma \tau} \qquad \mathsf{L}\mathbf{p}^{\mathrm{ss}} = 0$$

Optimal time estimation

A time estimator $\Theta(\boldsymbol{\sigma})$ is a function of the sequence $\boldsymbol{\sigma} = \sigma_0 \to \sigma_1 \to \cdots \to \sigma_N$

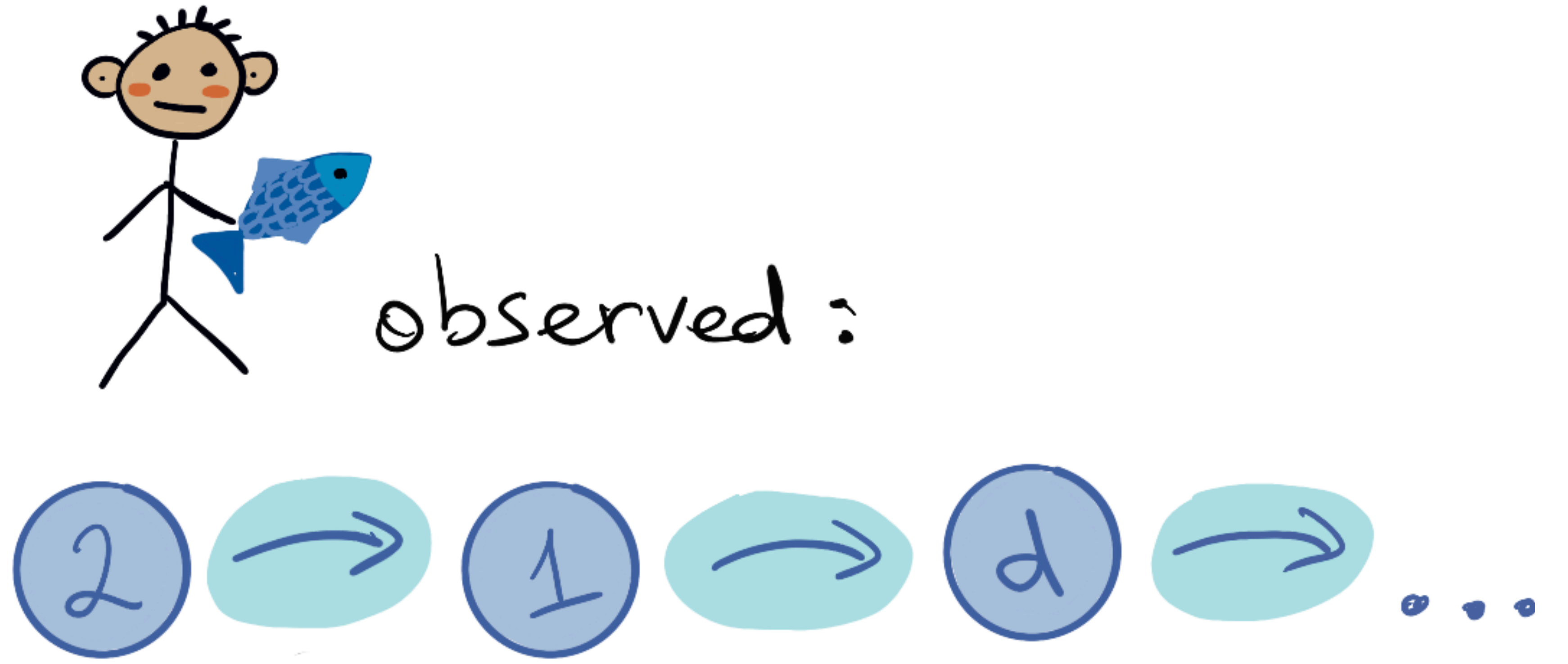
An **accurate** estimator is unbiased: $\mathrm{E}[\Theta] = \sum_{\boldsymbol{\sigma}} P(\boldsymbol{\sigma}\,|\,t)\Theta(\boldsymbol{\sigma}) = t$

A **precise** estimator has small variance $\mathrm{Var}[\Theta] = \mathrm{E}[\Theta^2] - \mathrm{E}[\Theta]^2$

An **efficient** estimator is unbiased and saturates the Cramér-Rao bound

$$\mathrm{Var}[\Theta]\mathscr{F}_t \geq 1$$

$\mathscr{F}_t = \mathrm{E}\left[\left(\frac{\partial \ln P(\boldsymbol{\sigma}\,|\,t)}{\partial t}\right)^2\right]$ is the **Fisher information**



Mean residual time

|T⏰CQS⟩

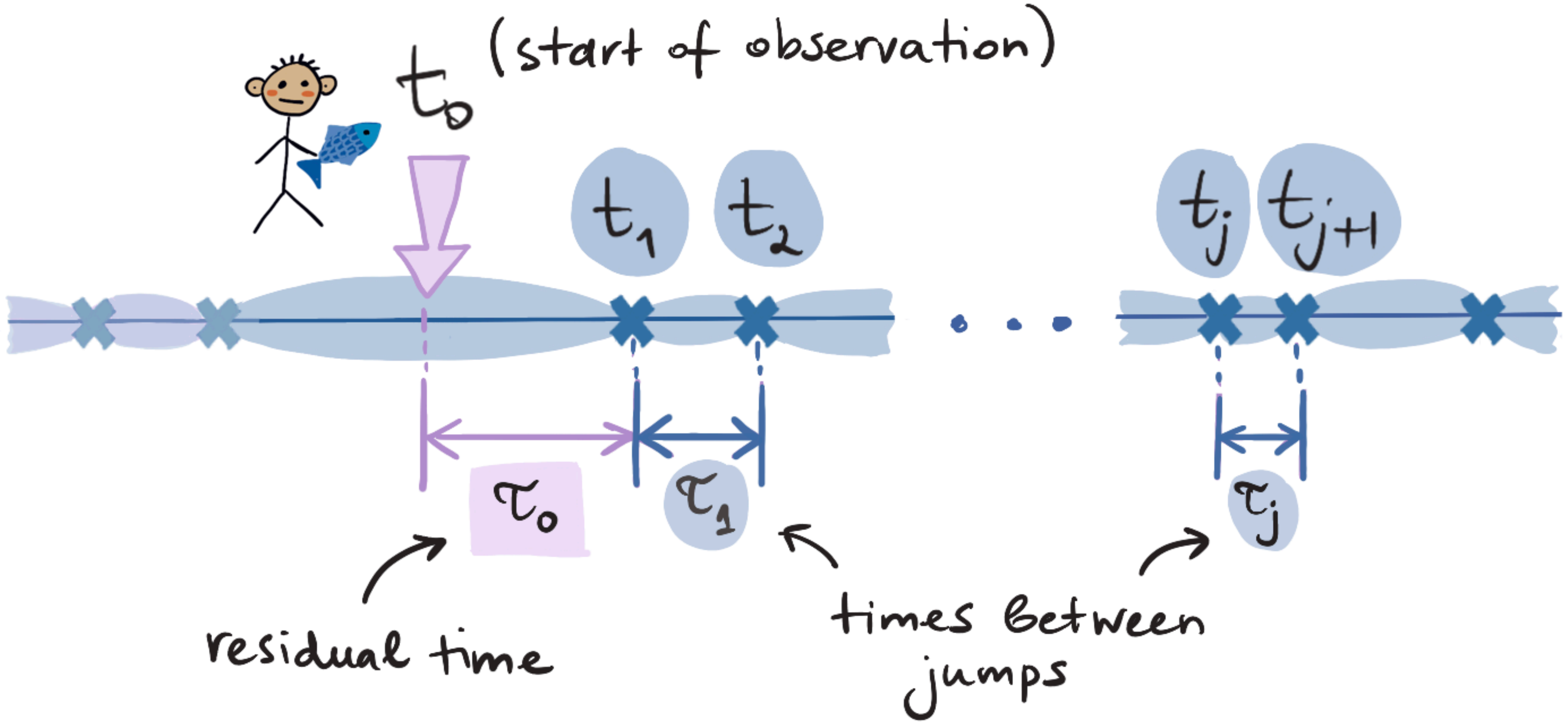
Fisher information in the saddle-point approximation:

$$\lim_{t\to\infty} t\mathcal{F}_t = \mathcal{T}^{-1}, \qquad \mathcal{T} = \sum_{\sigma=1}^{d} \frac{p_\sigma^{\mathrm{ss}}}{\Gamma_\sigma}$$

The **mean residual time** $\mathcal{T}$ controls the optimal asymptotic rate of information gain about time

$$\frac{\mathrm{E}[\Theta]^2}{\mathrm{Var}[\Theta]} \leq t\mathcal{T}^{-1}$$

(unbiased estimator & long times)



$$\mathcal{T} = \mathrm{E}[\tau_0] = \sum_\sigma p_\sigma^{\mathrm{ss}} \int_0^\infty d\tau\, W(\tau\,|\,\sigma)\tau$$

$$= \sum_\sigma p_\sigma^{\mathrm{ss}} \Gamma_\sigma^{-1}$$

Prech, Landi, Meier, Nurgalieva, Potts, Silva & MTM, arXiv:2406.19450

Inspection paradox

|T⏰CQS⟩

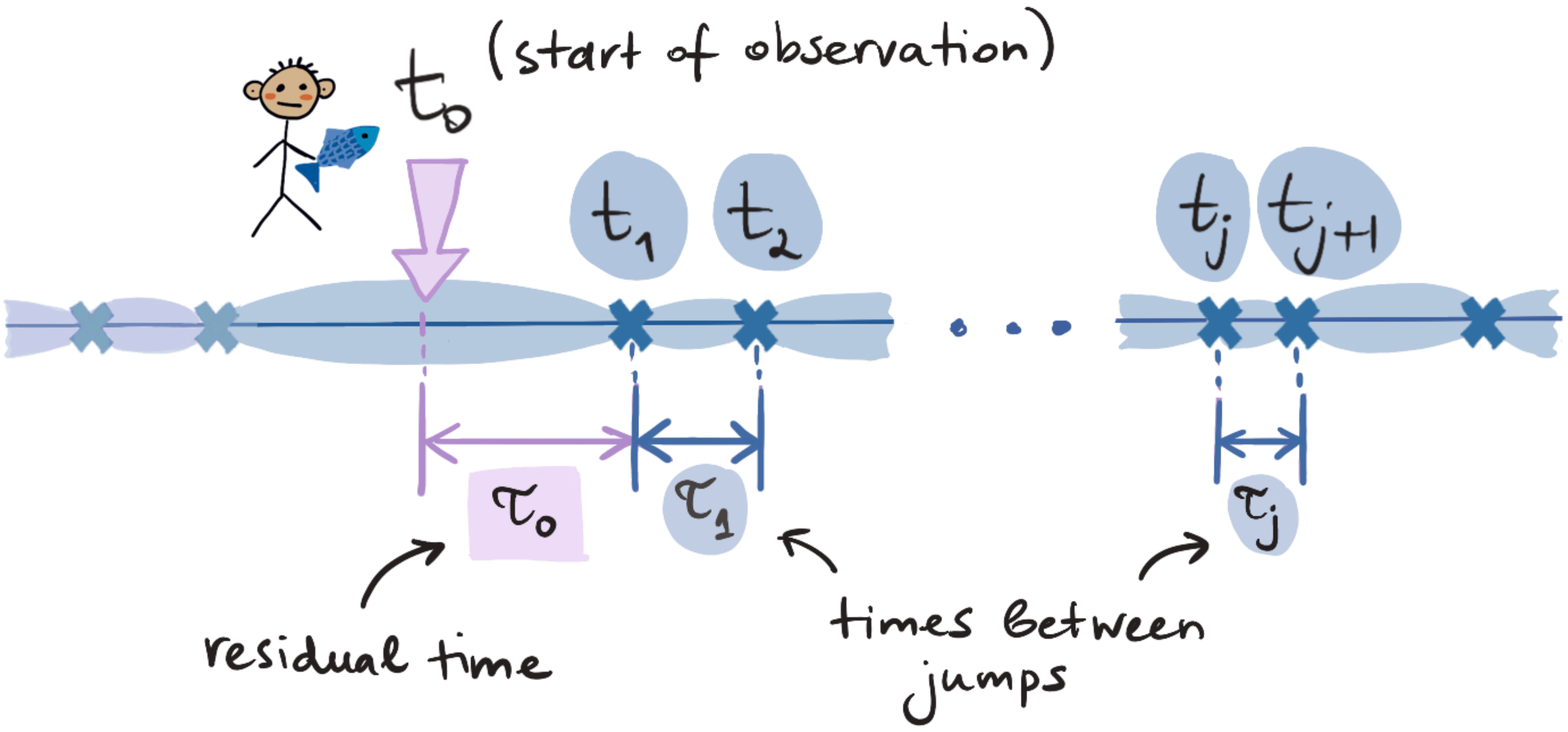
Mean residual time: $\mathrm{E}[\tau_0] = \mathcal{T} = \sum_\sigma p_\sigma^{\mathrm{ss}} \Gamma_\sigma^{-1}$.

Mean waiting time: $\mathrm{E}[\tau_j] = \mathcal{A}^{-1}$, where $\mathcal{A} = \sum_{\mu,\sigma} p_\sigma^{\mathrm{ss}} R_{\mu\sigma} = \sum_\sigma p_\sigma^{\mathrm{ss}} \Gamma_\sigma$ is the **dynamical activity**

Inspection paradox! $\mathcal{T} \geq \mathcal{A}^{-1}$

$\mathcal{T}$ is the *arithmetic* mean of Γ_σ^{-1}

$\mathcal{A}^{-1}$ is the *harmonic* mean of Γ_σ^{-1}



Clock uncertainty relation

$|\mathrm{T}\!\circ\!\mathrm{CQS}\rangle$

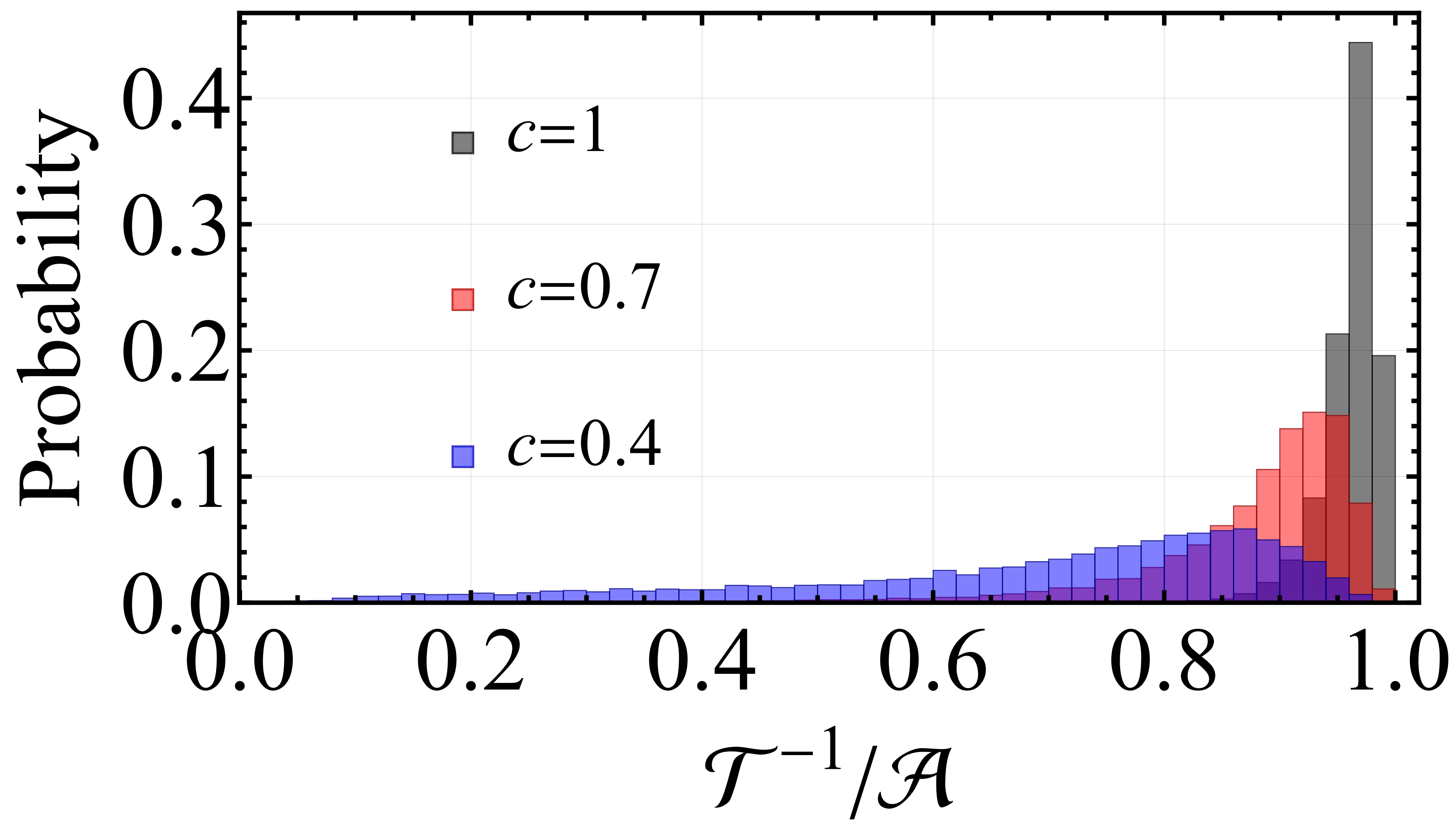
General CRB: $\mathrm{Var}[\Theta]\mathscr{F}_t \geq \left(\frac{\partial \mathrm{E}[\Theta]}{\partial t}\right)^2 \quad \Rightarrow \quad \boxed{\frac{(\partial_t \mathrm{E}[\Theta])^2}{\mathrm{Var}[\Theta]/t} \equiv \mathcal{S} \leq \mathscr{T}^{-1}}$

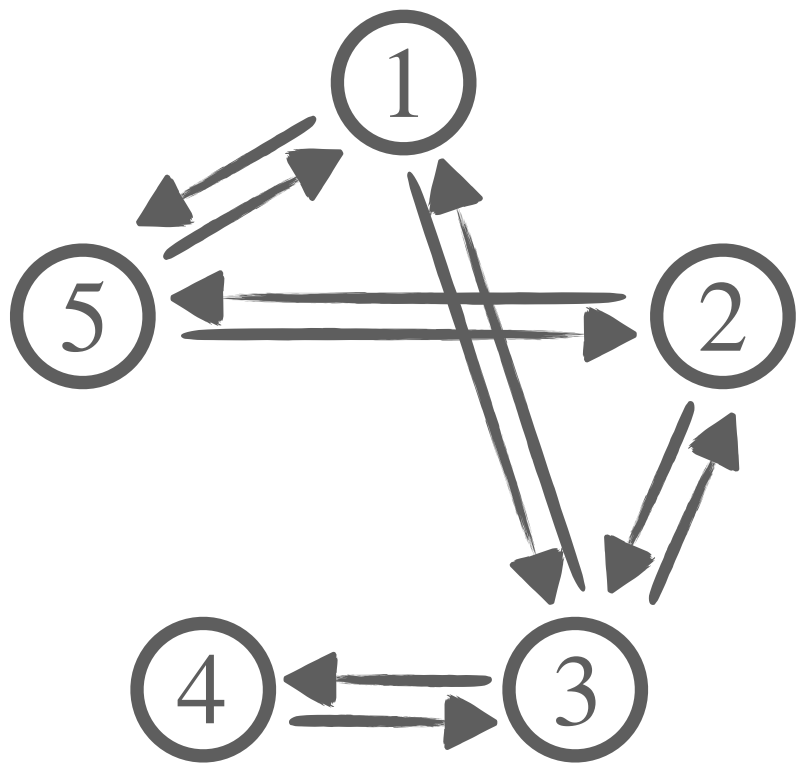
Bounds steady-state fluctuations of **arbitrary** observables $\Theta(\boldsymbol{\sigma})$ by measure of activity $\mathscr{T}^{-1}$

Non-trivial bound if $\mathrm{E}[\Theta] \sim t$, e.g. counting observables $\Theta(\boldsymbol{\sigma}) = \sum_{\mu,\sigma} w_{\mu\sigma} N_{\mu\sigma}$ ⟵ # of jumps $\sigma \to \mu$

arbitrary (real) weights

Tighter than the KUR since $\mathscr{T}^{-1} \leq \mathscr{A}$ (and always saturable)





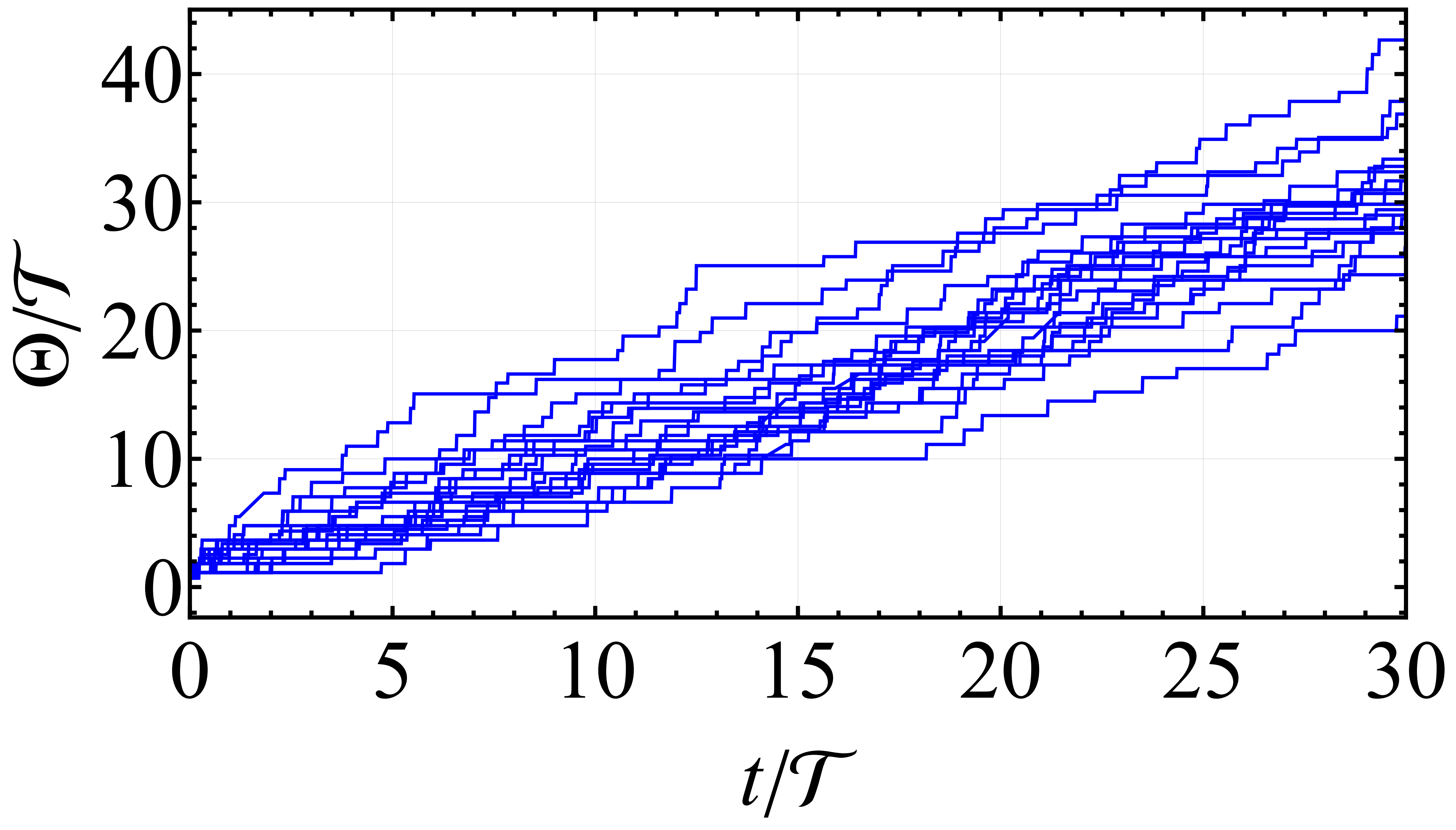
Prech, Landi, Meier, Nurgalieva, Potts, Silva & MTM, arXiv:2406.19450

Best unbiased linear estimator

|T⏰CQS⟩

Linear counting estimator $\Theta = \sum_{\mu,\sigma} w_{\mu\sigma} N_{\mu\sigma}$ ⟵ # of jumps $\sigma \to \mu$

arbitrary (real) weights



Mean current: $J = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}[\Theta] = \sum_{\mu,\sigma} w_{\mu\sigma} J_{\mu\sigma} \equiv \vec{w} \cdot \vec{J}$

Current noise: $D = \lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Var}[\Theta] = \vec{w} \cdot \mathbb{D} \cdot \vec{w}$

Best unbiased linear estimator (BLUE): minimise D subject to the unbiased constraint $J = 1$

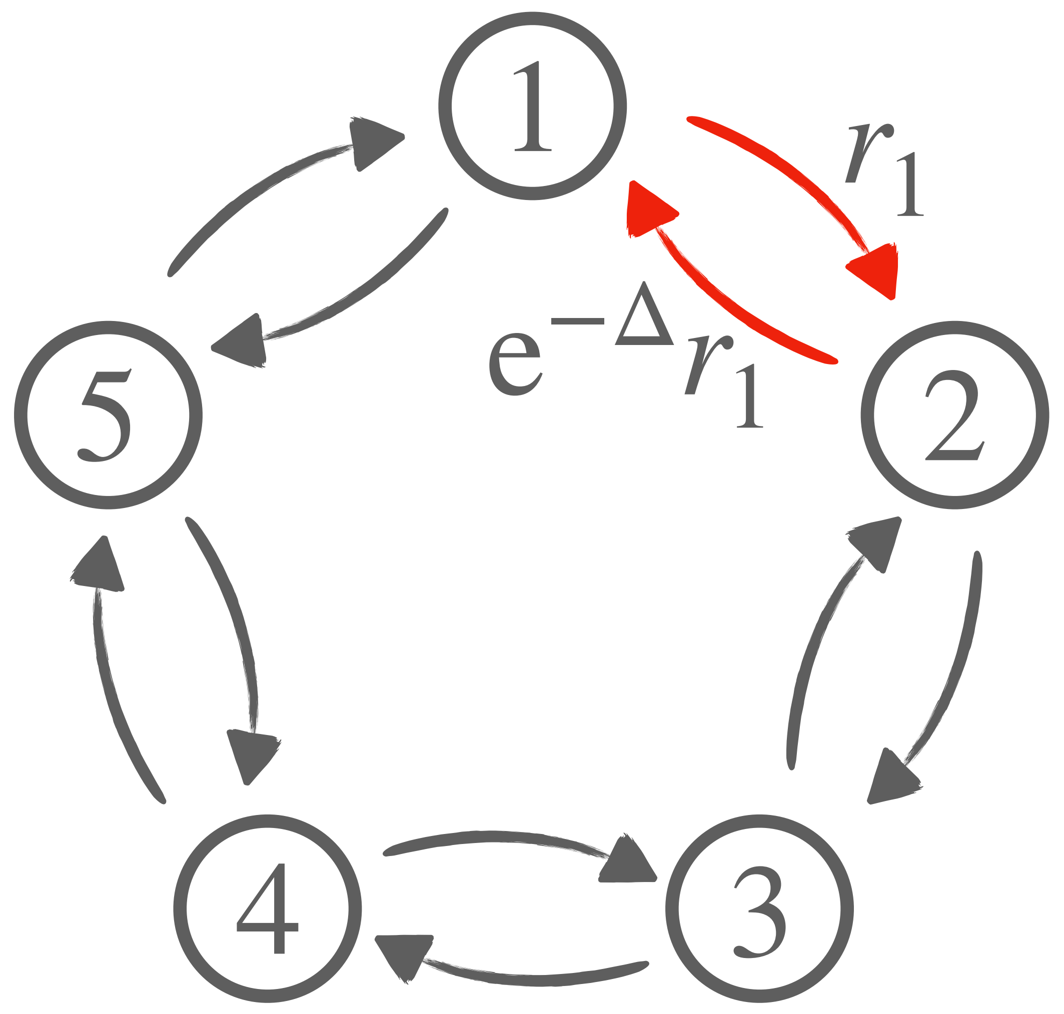
Equivalent to maximising the SNR $\mathcal{S} = \dfrac{(\vec{w} \cdot \vec{J})^2}{\vec{w} \cdot \mathbb{D} \cdot \vec{w}}$, since $\mathcal{S}$ is invariant under rescaling $\vec{w} \to r\vec{w}$

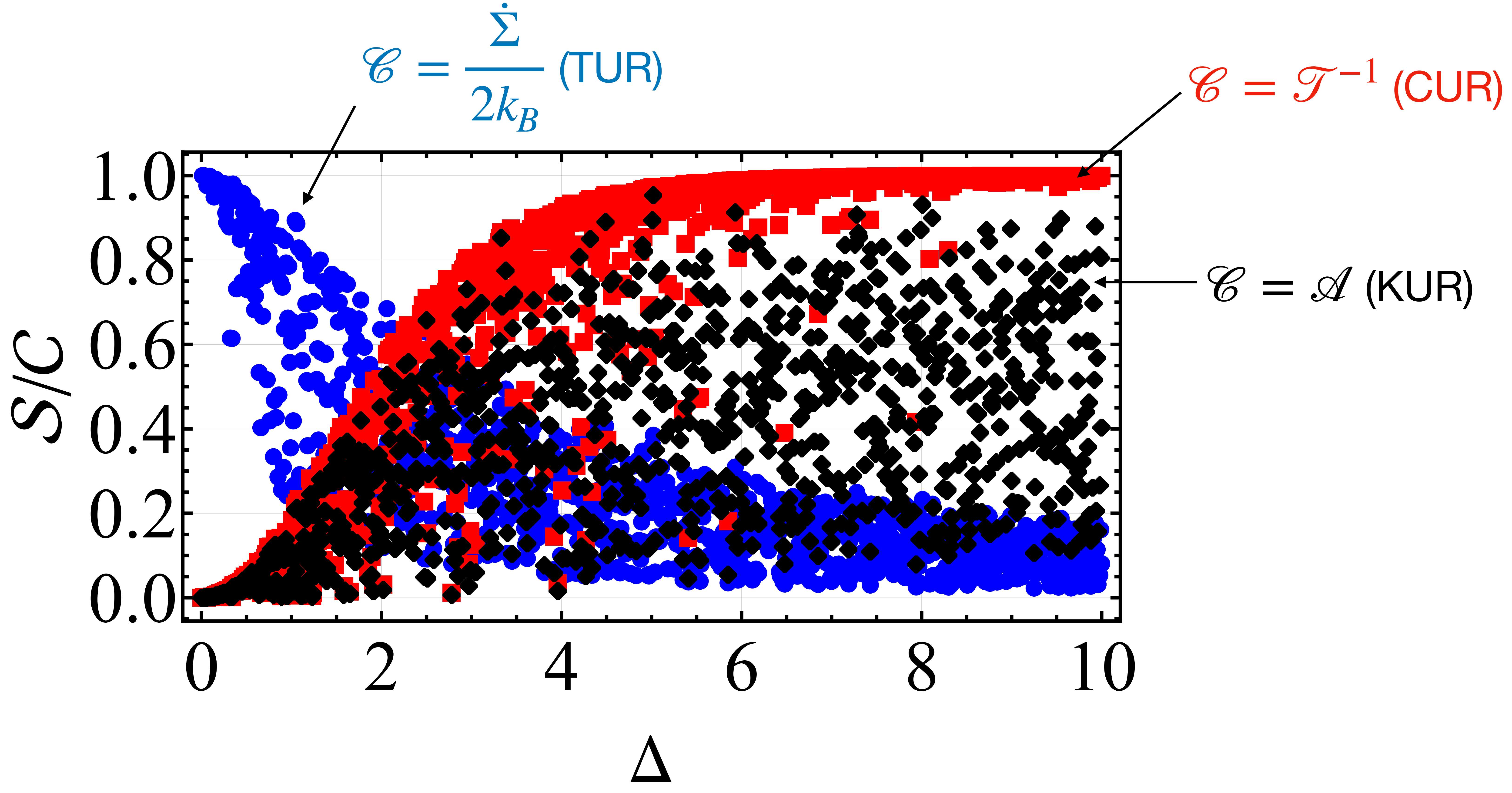
(One) solution is $w_{\mu\sigma} = \dfrac{1}{\Gamma_\sigma} \quad \Rightarrow \quad \mathcal{S} = \mathcal{T}^{-1}$ so the **CUR can always be saturated**!

Prech, Landi, Meier, Nurgalieva, Potts, Silva & MTM, arXiv:2406.19450

CUR is the tightest precision bound far from equilibrium

SNR ⟶ $\boxed{\mathscr{S} \leq \mathscr{C}}$ ⟵ cost





$\mathscr{T}^{-1}$ is the measure of activity that controls steady-state fluctuations far from equilibrium

Prech, Landi, Meier, Nurgalieva, Potts, Silva & MTM, arXiv:2406.19450

CUR is the tightest accuracy-resolution tradeoff

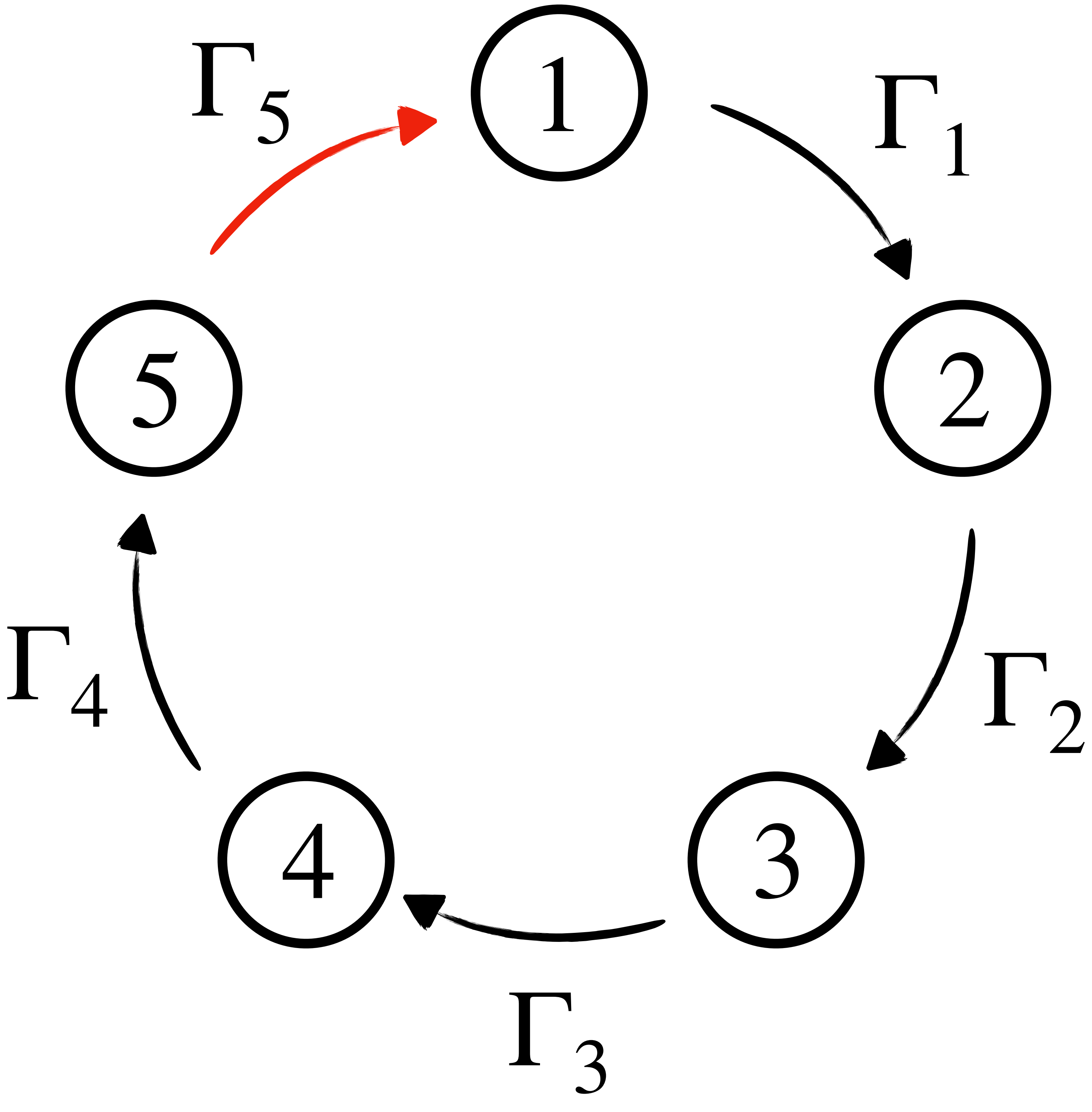
|T⏲CQS⟩

Accuracy-resolution tradeoff $\boxed{\mathcal{N}\nu \leq \mathcal{T}^{-1}}$ e.g. Erlang clock ($d = 5$ in the picture below)

BLUE counts every transition

Erlang estimator counts only one transition.

Estimator	Weights	SNR	Resolution	Accuracy
BLUE	$w_{\mu\sigma} = \Gamma_\sigma^{-1}$	$\mathcal{S} = \mathcal{T}^{-1}$	$\nu = \mathcal{A}$	$\mathcal{N} = 1/\mathcal{A}\mathcal{T}$
Erlang	$w_{1d} = J_{1d}^{-1}$	$\mathcal{S} = \mathcal{T}^{-1}$	$\nu = \mathcal{A}/d$	$\mathcal{N} = d/\mathcal{A}\mathcal{T}$



Maximum accuracy when $\Gamma_\sigma = \Gamma \Rightarrow \mathcal{N}_{\mathrm{Erl}} = d$

Woods, Silva, Pütz, Stupar & Renner, PRX Quantum **3**, 010319 (2022)

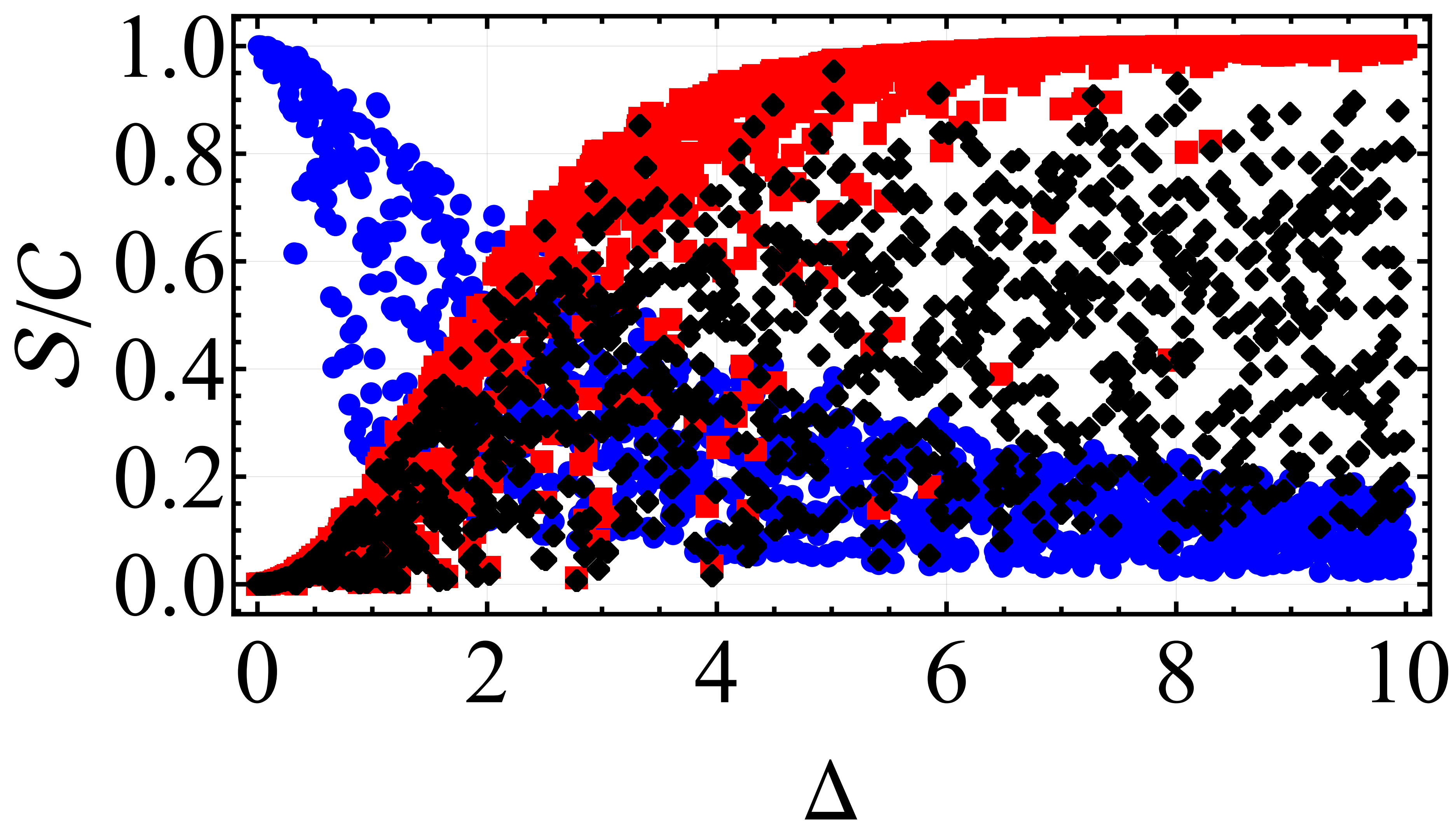
Prech, Landi, Meier, Nurgalieva, Potts, Silva & MTM, arXiv:2406.19450

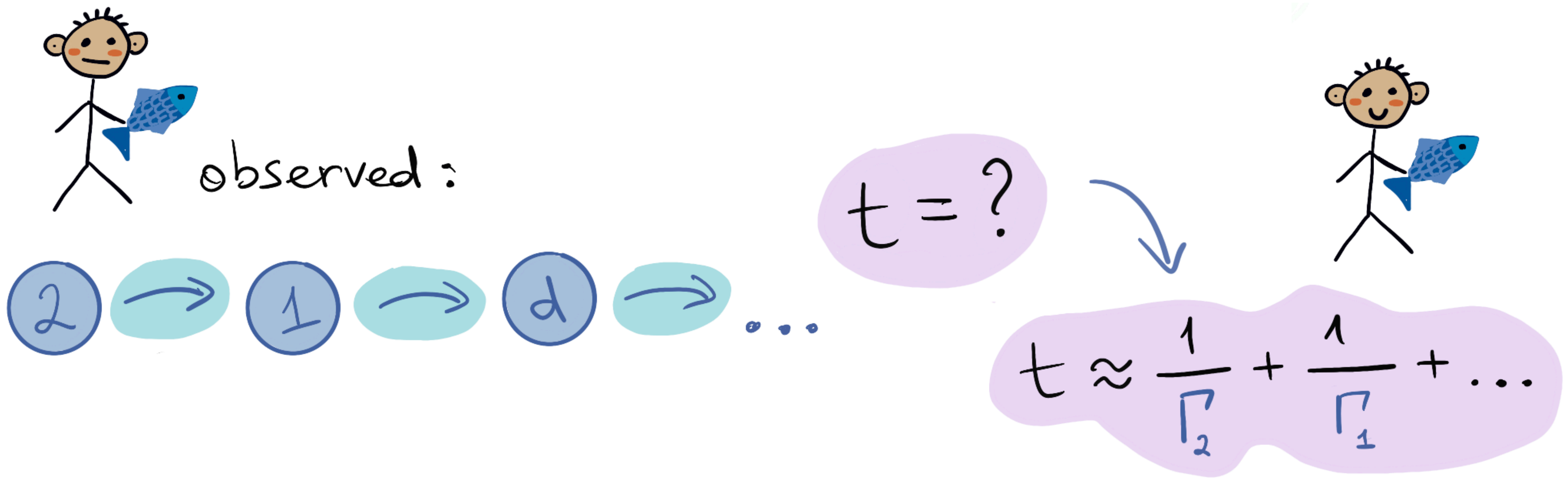
Summary

- **Clock uncertainty relation** $\mathscr{S} \leq \mathscr{T}^{-1}$: a kinetic precision bound that is **always saturable**

- **Mean residual time** $\mathscr{T} = \sum_\sigma p_\sigma^{\mathrm{ss}} \Gamma_\sigma^{-1}$ as a useful measure of activity in stochastic thermodynamics?

- **Optimal time estimator** has interesting features:
  - Weights each jump by its mean waiting time (intuitive)
  - Is time-reversal invariant (surprising)
  - Has delta-correlated fluctuations (very surprising!)





- Signatures of quantum/non-Markov dynamics? Quantum clock precision?

Prech et al., arXiv:2406.19450
Macieszczak, arXiv:2407.09839

Big picture

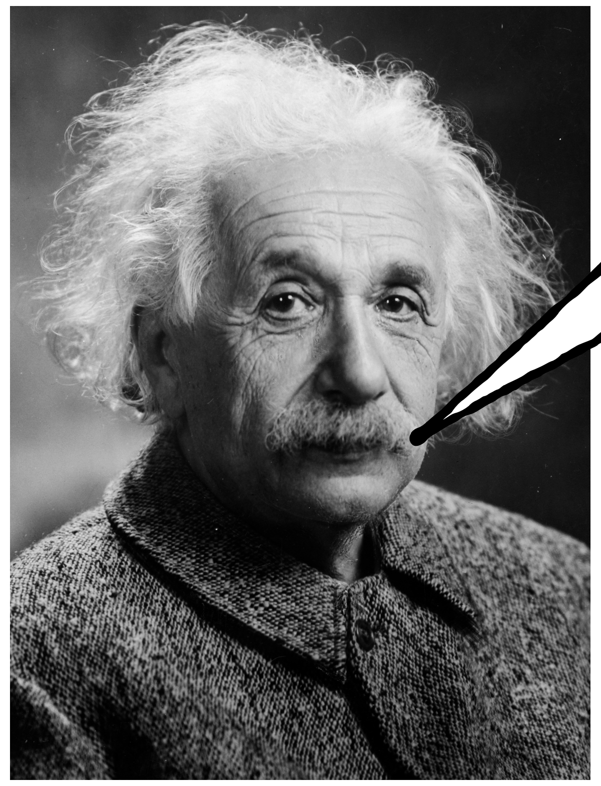
|T⏰CQS⟩

Thermodynamic and kinetic constraints on clocks:

- accurate clocks dissipate a lot of energy
- tradeoff between entropy production, accuracy, and resolution

So what?

- Important when extreme precision (e.g. quantum computation, fundamental physics) or low dissipation (e.g. space applications) is important
- Evidence for extreme (e.g. exponential) quantum precision enhancement
- What is time?



Coming soon:

- Observing these tradeoffs in electronic devices
- Implications for thermodynamics of other continuous measurements

|T🕒CQS⟩

Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Thank you

Kacper Prech
(Basel)

Gabriel Landi
(Rochester)

Florian Meier
(TU Vienna)

Nuriya Nurgalieva
(ETH Zurich)

Patrick Potts
(Basel)

Ralph Silva
(ETH Zurich)

arXiv:2406.19450

sfi Fondúireacht Eolaíochta Éireann
Dá bhfuil romhainn
Science Foundation Ireland
For what's next

UK Research
and Innovation

THE
ROYAL
SOCIETY

Co-funded by the
European Union

ASPECTS
aspects-quantum.com