# Controllability of Schrödinger equations and application to quantum rotors

Eugenio Pozzoli (CNRS, IRMAR, Université de Rennes, France)

24-26 February 2025, Dijon - *Winter school on BEC*

1: Controllability of finite-dimensional Schrödinger equations
2: Spectral conditions and resonant control
3: Controllability of $\infty$-dimensional Schrödinger equations

# Controllability of finite-dimensional Schrödinger equations

1. An algebraic condition for controllability
2. And its proof
3. An algebraic condition for small-time controllability
4. Small-time controllability of scalar-input systems

# The Schrödinger equation for the state and for the propagator

***Controlled quantum evolution*** for the state
$\psi(t) \in \mathcal{S}^N = \{\psi \in \mathbb{C}^N \mid |\psi| = 1\}$ (it has real dimension $2N - 1$)

$$i\frac{d}{dt}\psi(t) = \left(H_0 + \sum_{j=1}^{m} u_j(t) H_j\right)\psi(t), \quad \psi(t=0) = \psi_0 \in \mathcal{S}^N.$$

# The Schrödinger equation for the state and for the propagator

***Controlled quantum evolution*** for the state
$\psi(t) \in \mathcal{S}^N = \{\psi \in \mathbb{C}^N \mid |\psi| = 1\}$ (it has real dimension $2N-1$)

$$i\frac{d}{dt}\psi(t) = \left(H_0 + \sum_{j=1}^{m} u_j(t) H_j\right)\psi(t), \quad \psi(t=0) = \psi_0 \in \mathcal{S}^N.$$

- ***The generators***: $H_0$ (the drift), $H_1, \ldots, H_m$ (the interactions), ***Hermitian matrices*** on $\mathcal{H} = \mathbb{C}^N$: $H_j^* = H_j$. Alternatively, $(iH_j)^* = -iH_j$, i.e. $H_j \in \mathfrak{u}(N)$. (It has real dim. $N^2$)

# The Schrödinger equation for the state and for the propagator

***Controlled quantum evolution*** for the state $\psi(t) \in \mathcal{S}^N = \{\psi \in \mathbb{C}^N \mid |\psi| = 1\}$ (it has real dimension $2N-1$)

$$i\frac{d}{dt}\psi(t) = \left(H_0 + \sum_{j=1}^{m} u_j(t) H_j\right)\psi(t), \quad \psi(t=0) = \psi_0 \in \mathcal{S}^N.$$

- ***The generators***: $H_0$ (the drift), $H_1, \ldots, H_m$ (the interactions), ***Hermitian matrices*** on $\mathcal{H} = \mathbb{C}^N$: $H_j^* = H_j$. Alternatively, $(iH_j)^* = -iH_j$, i.e. $H_j \in \mathfrak{u}(N)$. (It has real dim. $N^2$)
- ***The evolutions***: $e^{itH_j}, t \in \mathbb{R}$, ***unitary matrices*** on $\mathcal{H} = \mathbb{C}^N$: $(e^{itH_j})^* = (e^{itH_j})^{-1} = e^{-itH_j}$. I.e., $e^{itH_j} \in U(N)$. In particular, $|e^{itH_j}\psi| = |\psi| = 1$.

# The Schrödinger equation for the state and for the propagator

***Controlled quantum evolution*** for the state
$\psi(t) \in \mathcal{S}^N = \{\psi \in \mathbb{C}^N \mid |\psi| = 1\}$ (it has real dimension $2N - 1$)

$$i \frac{d}{dt}\psi(t) = \left( H_0 + \sum_{j=1}^{m} u_j(t) H_j \right) \psi(t), \quad \psi(t = 0) = \psi_0 \in \mathcal{S}^N.$$

- ***The generators***: $H_0$ (the drift), $H_1, \dots, H_m$ (the interactions), ***Hermitian matrices*** on $\mathcal{H} = \mathbb{C}^N$: $H_j^* = H_j$. Alternatively, $(iH_j)^* = -iH_j$, i.e. $H_j \in \mathfrak{u}(N)$. (It has real dim. $N^2$)
- ***The evolutions***: $e^{itH_j}$, $t \in \mathbb{R}$, ***unitary matrices*** on $\mathcal{H} = \mathbb{C}^N$: $(e^{itH_j})^* = (e^{itH_j})^{-1} = e^{-itH_j}$. I.e., $e^{itH_j} \in U(N)$. In particular, $|e^{itH_j}\psi| = |\psi| = 1$.

***Controlled quantum evolution*** for the propagator $U \in U(N)$:

$$i \frac{d}{dt}U(t) = \left( H_0 + \sum_{j=1}^{m} u_j(t) H_j \right) U(t), \quad U(t = 0) = \mathrm{I}.$$

# The controllability problem

- To every initial state $\psi_0$ and ***control function*** $u : [0, T] \to \mathbb{R}^m$, we associate the wavefunction $\psi(t, u, \psi_0)$ at time $t \in [0, T]$, that is, the solution of the ODE.
- E.g., if $u$ is piecewise constant, then

$$\psi(t, u, \psi_0) = e^{i(t-\sum_{j=1}^{k} t_j)(H_0+\sum_{j=1}^{m} u_j(t_k)H_j)} \ldots e^{it_1(H_0+\sum_{j=1}^{m} u_j(t_1)H_j)}\psi_0.$$

**The controllability problem**

Given $\psi_0, \psi_1 \in \mathcal{S}^n$, find $u : [0, T] \to R^m$ such that $\psi(T, u, \psi_0) = \psi_1$.

# The controllability problem

- To every initial state $\psi_0$ and ***control function*** $u : [0, T] \to \mathbb{R}^m$, we associate the wavefunction $\psi(t, u, \psi_0)$ at time $t \in [0, T]$, that is, the solution of the ODE.
- E.g., if $u$ is piecewise constant, then

$$\psi(t, u, \psi_0) = e^{i(t - \sum_{j=1}^{k} t_j)(H_0 + \sum_{j=1}^{m} u_j(t_k) H_j)} \ldots e^{i t_1 (H_0 + \sum_{j=1}^{m} u_j(t_1) H_j)} \psi_0 .$$

## The controllability problem

Given $\psi_0, \psi_1 \in \mathcal{S}^n$, find $u : [0, T] \to R^m$ such that $\psi(T, u, \psi_0) = \psi_1$.

## The approximate controllability problem

Given $\psi_0, \psi_1 \in \mathcal{S}^n$, and an error $\varepsilon > 0$, find $u : [0, T] \to R^m$ such that $|\psi(T, u, \psi_0) - \psi_1| < \varepsilon$.

We say that the equation is (approximately) controllable if the (approximate) controllability problem is solvable for every $\psi_0, \psi_1 \in \mathcal{S}^n$.

# Lie algebras

The vector space of anti-Hermitian (traceless) $N \times N$ complex matrices $\mathfrak{u}(N)$ ($\mathfrak{su}(N)$) is a ***Lie algebra***: if $A, B \in \mathfrak{u}(n)$, then $[A, B] := AB - BA \in \mathfrak{u}(N)$. Given $A_1, \ldots, A_m \in \mathfrak{u}(N)$, we introduce

$$\mathrm{Lie}\{A_1, \ldots, A_m\} \subset \mathfrak{u}(N)$$

defined as the smallest vector space containing $A_1, \ldots, A_m$, closed under commutator:

$$\mathcal{C}, \mathcal{D} \in \mathrm{Lie}\{A_1, \ldots, A_m\} \Rightarrow [\mathcal{C}, \mathcal{D}] := \mathcal{C}\mathcal{D} - \mathcal{D}\mathcal{C} \in \mathrm{Lie}\{A_1, \ldots, A_m\}.$$

# Lie algebras

The vector space of anti-Hermitian (traceless) $N \times N$ complex matrices $\mathfrak{u}(N)$ ($\mathfrak{su}(N)$) is a ***Lie algebra***: if $A, B \in \mathfrak{u}(n)$, then $[A, B] := AB - BA \in \mathfrak{u}(N)$. Given $A_1, \ldots, A_m \in \mathfrak{u}(N)$, we introduce

$$\mathrm{Lie}\{A_1, \ldots, A_m\} \subset \mathfrak{u}(N)$$

defined as the smallest vector space containing $A_1, \ldots, A_m$, closed under commutator:

$$\mathcal{C}, \mathcal{D} \in \mathrm{Lie}\{A_1, \ldots, A_m\} \Rightarrow [\mathcal{C}, \mathcal{D}] := \mathcal{C}\mathcal{D} - \mathcal{D}\mathcal{C} \in \mathrm{Lie}\{A_1, \ldots, A_m\}.$$

Example: The Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

form a basis of $\mathfrak{su}(2)$. They satisfy the commutation relations

$$[\sigma_z, \sigma_x] = 2\sigma_y, \quad [\sigma_y, \sigma_z] = 2\sigma_x, \quad [\sigma_x, \sigma_y] = 2\sigma_z.$$

# A criterium for controllability

**Theorem**

The Schrödinger equation for the propagator is controllable iff $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$. In particular, if $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$ the Schrödinger equation for the state is controllable.

[1] Jurdjevic, Sussmann; Control Systems on Lie Groups. J. Diff. Eq. 12, 313-329 (1972)

[2] Albertini, D'Alessandro; Notions of Controllability for Quantum Mechanical Systems. Proceedings of the 40th IEEE Conference on Decision and Control (2001)

# A criterium for controllability

**Theorem**

The Schrödinger equation for the propagator is controllable iff $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$. In particular, if $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$ the Schrödinger equation for the state is controllable.

In the rest of this lecture, we prove a weaker version, namely: if $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$, the eq. for the state is approx. controllable.

[1] Jurdjevic, Sussmann; Control Systems on Lie Groups. J. Diff. Eq. 12, 313-329 (1972)

[2] Albertini, D'Alessandro; Notions of Controllability for Quantum Mechanical Systems. Proceedings of the 40th IEEE Conference on Decision and Control (2001)

# A criterium for controllability

## Theorem

The Schrödinger equation for the propagator is controllable iff $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$. In particular, if $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$ the Schrödinger equation for the state is controllable.

In the rest of this lecture, we prove a weaker version, namely: if $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$, the eq. for the state is approx. controllable.

- The result (for the propagator) is true more in general[1] for any compact connected Lie group $G$, which in this case is $U(n)$.
- The converse statement (for the state) is also true[2] when $N$ is odd (with $\mathfrak{su}(N)$ instead of $\mathfrak{u}(N)$), but not when $N$ is even (there exist proper subgroups of $SU(N)$ acting transitively on the sphere $\mathcal{S}^N$).

[1] Jurdjevic, Sussmann; Control Systems on Lie Groups. J. Diff. Eq. 12, 313-329 (1972)

[2] Albertini, D'Alessandro; Notions of Controllability for Quantum Mechanical Systems. Proceedings of the 40th IEEE Conference on Decision and Control (2001)

# Structure of the proof

We show that:

- For any $\psi_0, \psi_1 \in \mathcal{S}^N$, there exists $U \in U(N)$ such that $U\psi_0 = \psi_1$ (i.e., the action of $U(N)$ on $\mathcal{S}^N$ is ***transitive***).

# Structure of the proof

We show that:

- For any $\psi_0, \psi_1 \in \mathcal{S}^N$, there exists $U \in U(N)$ such that $U\psi_0 = \psi_1$ (i.e., the action of $U(N)$ on $\mathcal{S}^N$ is ***transitive***).
- For any $U \in U(n)$ there exists $A \in \mathfrak{u}(n)$ such that $U = e^A$ (i.e., the exponential map $\exp : \mathfrak{u}(N) \to U(N)$ is ***surjective***).

# Structure of the proof

We show that:

- For any $\psi_0, \psi_1 \in \mathcal{S}^N$, there exists $U \in U(N)$ such that $U\psi_0 = \psi_1$ (i.e., the action of $U(N)$ on $\mathcal{S}^N$ is ***transitive***).
- For any $U \in U(n)$ there exists $A \in \mathfrak{u}(n)$ such that $U = e^A$ (i.e., the exponential map $\exp : \mathfrak{u}(N) \to U(N)$ is ***surjective***).
- For any $A \in \mathfrak{u}(n)$, we can approximately control the system from $\psi_0$ towards $e^A\psi_0$.

# $U(N) \circlearrowright \mathcal{S}^N$ is transitive

Given $\psi_0$, complete it to an orthonormal basis $\{\psi_0, v_2, \ldots, v_N\}$ of $\mathbb{C}^N$.

# $U(N) \circlearrowright \mathcal{S}^N$ is transitive

Given $\psi_0$, complete it to an orthonormal basis $\{\psi_0, v_2, \ldots, v_N\}$ of $\mathbb{C}^N$.The matrix made of columns

$$V = (\psi_0, v_2, \ldots, v_N) \in U(N)$$

and is such that $Ve_1 = \psi_0$.

# $U(N) \circlearrowright \mathcal{S}^N$ is transitive

Given $\psi_0$, complete it to an orthonormal basis $\{\psi_0, v_2, \ldots, v_N\}$ of $\mathbb{C}^N$.The matrix made of columns

$$V = (\psi_0, v_2, \ldots, v_N) \in U(N)$$

and is such that $Ve_1 = \psi_0$.In the same way, we can construct $W \in U(N)$ such that $We_1 = \psi_1$.

# $U(N) \circlearrowright \mathcal{S}^N$ is transitive

Given $\psi_0$, complete it to an orthonormal basis $\{\psi_0, v_2, \ldots, v_N\}$ of $\mathbb{C}^N$.The matrix made of columns

$$V = (\psi_0, v_2, \ldots, v_N) \in U(N)$$

and is such that $Ve_1 = \psi_0$.In the same way, we can construct $W \in U(N)$ such that $We_1 = \psi_1$.Hence, $U := WV^* \in U(N)$ is such that

$$U\psi_0 = WV^*\psi_0 = We_1 = \psi_1.$$

# $\exp : \mathfrak{u}(N) \to U(N)$ is surjective

By the spectral theorem for normal matrices, diagonalize $U = VD_U V^*$, where $V \in U(N)$ and

$$D_U = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N}), \quad \theta_i \in \mathbb{R}.$$

# $\exp : \mathfrak{u}(N) \to U(N)$ is surjective

By the spectral theorem for normal matrices, diagonalize $U = VD_U V^*$, where $V \in U(N)$ and

$$D_U = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_N}), \quad \theta_i \in \mathbb{R}.$$

Notice that

$$D_U = \exp(D_A), \quad D_A = \mathrm{diag}(i\theta_1, \dots, i\theta_N).$$

# $\exp : \mathfrak{u}(N) \to U(N)$ is surjective

By the spectral theorem for normal matrices, diagonalize $U = VD_U V^*$, where $V \in U(N)$ and

$$D_U = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N}), \quad \theta_i \in \mathbb{R}.$$

Notice that

$$D_U = \exp(D_A), \quad D_A = \mathrm{diag}(i\theta_1, \ldots, i\theta_N).$$

Then,

$$A := VD_A V^* \in \mathfrak{u}(N), \quad e^A = U.$$

# Recurrent vector fields

## Flow of a vector field

Given a smooth vector field $f : M \to TM$ we denote by $\phi_f^t$ its flow at time $t$, that is, $x(t) := \phi_f^t(x_0)$ solves the ODE

$$\frac{d}{dt}x(t) = f(x(t)), \quad x(0) = x_0.$$

# Recurrent vector fields

## Flow of a vector field

Given a smooth vector field $f : M \to TM$ we denote by $\phi_f^t$ its flow at time $t$, that is, $x(t) := \phi_f^t(x_0)$ solves the ODE

$$\frac{d}{dt}x(t) = f(x(t)), \quad x(0) = x_0.$$

## Recurrent vector field (e.g., periodic vector fields)

A vector field $f : M \to TM$ is recurrent if for every $x \in M$, ngbhd $V_x$, and time $t > 0$, there exists $T \geqslant t$ such that $\phi_f^T(V_x) \cap V_x \neq \emptyset$.

# Recurrent vector fields

## Flow of a vector field

Given a smooth vector field $f : M \to TM$ we denote by $\phi_f^t$ its flow at time $t$, that is, $x(t) := \phi_f^t(x_0)$ solves the ODE

$$\frac{d}{dt}x(t) = f(x(t)), \quad x(0) = x_0.$$

## Recurrent vector field (e.g., periodic vector fields)

A vector field $f : M \to TM$ is recurrent if for every $x \in M$, ngbhd $V_x$, and time $t > 0$, there exists $T \geqslant t$ such that $\phi_f^T(V_x) \cap V_x \neq \emptyset$.

## Lemma 1

If a vector field $f$ is recurrent, then for every $t > 0$,

$$\phi_f^{-t}(x_0) \in \overline{\{\phi_f^s(x_0), s \geqslant 0\}}.$$

By recurrence, there exists $s_k \uparrow \infty$ such that $\phi_f^{s_k}(\phi_f^{-t}(x_0)) \to \phi_f^{-t}(x_0)$.

# A theorem of Poincaré

## Poincaré's Theorem

Let $M$ be compact with finite volume $\mathrm{vol}(M) < \infty$. If the flow of a vector field $f$ preserves the volumes, the vector field is recurrent.

# A theorem of Poincaré

## Poincaré's Theorem

Let $M$ be compact with finite volume $\mathrm{vol}(M) < \infty$. If the flow of a vector field $f$ preserves the volumes, the vector field is recurrent.

Let $x \in M$, $V_x$, $t > 0$. Consider the sets

$$\phi_f^{nt}(V_x), n \in \mathbb{N}.$$

# A theorem of Poincaré

## Poincaré's Theorem

Let $M$ be compact with finite volume $\mathrm{vol}(M) < \infty$. If the flow of a vector field $f$ preserves the volumes, the vector field is recurrent.

Let $x \in M$, $V_x$, $t > 0$. Consider the sets

$$\phi_f^{nt}(V_x), n \in \mathbb{N}.$$

Since the flow preserves the volume, and the manifold has finite volume, it follows that there exist $n, m \in \mathbb{N}, n > m$, such that

$$\phi_f^{nt}(V_x) \cap \phi_f^{mt}(V_x) \neq \emptyset.$$

# A theorem of Poincaré

**Poincaré's Theorem**

Let $M$ be compact with finite volume $\mathrm{vol}(M) < \infty$. If the flow of a vector field $f$ preserves the volumes, the vector field is recurrent.

Let $x \in M$, $V_x$, $t > 0$. Consider the sets

$$\phi_f^{nt}(V_x), n \in \mathbb{N}.$$

Since the flow preserves the volume, and the manifold has finite volume, it follows that there exist $n, m \in \mathbb{N}, n > m$, such that

$$\phi_f^{nt}(V_x) \cap \phi_f^{mt}(V_x) \neq \varnothing.$$

This implies (by applying $\phi_f^{-mt}$)

$$\phi_f^{(n-m)t}(V_x) \cap V_x \neq \varnothing.$$

Since $(n-m)t \geqslant t$, the proof is concluded.

# Schrödinger flows are volume preserving

- Given $A \in \mathfrak{u}(N)$, the flow $e^{tA}$ preserves the volume:

$$\mathrm{vol}(e^{tA}(V)) = \int_{e^{tA}(V)} d\psi \underbrace{=}_{\phi = e^{tA}\psi} \int_V \underbrace{|\det e^{-tA}|}_{=1} d\phi = \int_V d\phi = \mathrm{vol}(V).$$

# Schrödinger flows are volume preserving

- Given $A \in \mathfrak{u}(N)$, the flow $e^{tA}$ preserves the volume:

$$\mathrm{vol}(e^{tA}(V)) = \int_{e^{tA}(V)} d\psi \underbrace{=}_{\phi = e^{tA}\psi} \int_V \underbrace{|\det e^{-tA}|}_{=1} d\phi = \int_V d\phi = \mathrm{vol}(V).$$

- Since the sphere $\mathcal{S}^N$ is compact and of finite volume, Poincaré Theorem implies that the vector field $\psi \mapsto A\psi$ is recurrent.

# Schrödinger flows are volume preserving

- Given $A \in \mathfrak{u}(N)$, the flow $e^{tA}$ preserves the volume:

$$\mathrm{vol}(e^{tA}(V)) = \int_{e^{tA}(V)} d\psi \underbrace{=}_{\phi = e^{tA}\psi} \int_V \underbrace{|\det e^{-tA}|}_{=1} d\phi = \int_V d\phi = \mathrm{vol}(V).$$

- Since the sphere $\mathcal{S}^N$ is compact and of finite volume, Poincaré Theorem implies that the vector field $\psi \mapsto A\psi$ is recurrent.
- Lemma 1 implies that for every $\psi \in \mathcal{S}^N$, $t > 0$, we can approximately reach the state $e^{-itH_0}\psi$.

# How to reach the flow of Lie brackets?

## Lemma 2

For every $A, B$ matrices, we have

$$\left(e^{\frac{-A}{sn}} e^{-sB} e^{\frac{A}{sn}} e^{sB}\right)^n \xrightarrow[n\to\infty]{} \exp\left(-\frac{A}{s} + e^{-sB}\frac{A}{s}e^{sB}\right) \xrightarrow[s\to 0]{} e^{[A,B]}.$$

# How to reach the flow of Lie brackets?

## Lemma 2

For every $A, B$ matrices, we have

$$\left(e^{\frac{-A}{sn}} e^{-sB} e^{\frac{A}{sn}} e^{sB}\right)^n \xrightarrow[n\to\infty]{} \exp\left(-\frac{A}{s} + e^{-sB}\frac{A}{s}e^{sB}\right) \xrightarrow[s\to 0]{} e^{[A,B]}.$$

Ingredients:

- $e^{-C}e^{D}e^{C} = \exp\left(e^{-C}De^{C}\right)$;

# How to reach the flow of Lie brackets?

## Lemma 2

For every $A, B$ matrices, we have

$$\left(e^{\frac{-A}{sn}} e^{-sB} e^{\frac{A}{sn}} e^{sB}\right)^n \xrightarrow[n\to\infty]{} \exp\left(-\frac{A}{s} + e^{-sB}\frac{A}{s}e^{sB}\right) \xrightarrow[s\to 0]{} e^{[A,B]}.$$

Ingredients:

- $e^{-C}e^{D}e^{C} = \exp\left(e^{-C}De^{C}\right)$;
- $(e^{D/n}e^{C/n})^n \xrightarrow[n\to\infty]{} e^{D+C}$ (Lie-Trotter product formula);

# How to reach the flow of Lie brackets?

## Lemma 2

For every $A, B$ matrices, we have

$$\left(e^{\frac{-A}{sn}} e^{-sB} e^{\frac{A}{sn}} e^{sB}\right)^n \xrightarrow[n \to \infty]{} \exp\left(-\frac{A}{s} + e^{-sB}\frac{A}{s}e^{sB}\right) \xrightarrow[s \to 0]{} e^{[A,B]}.$$

Ingredients:

- $e^{-C} e^{D} e^{C} = \exp\left(e^{-C} D e^{C}\right)$;
- $(e^{D/n} e^{C/n})^n \xrightarrow[n \to \infty]{} e^{D+C}$ (Lie-Trotter product formula);
- $e^{-C} D e^{C} = \sum_{k=0}^{\infty} \frac{\mathrm{ad}_C^k(D)}{k!}$ where

$$\mathrm{ad}_C D = [D, C], \quad \mathrm{ad}_C^k D = [\mathrm{ad}_C^{k-1} D, C].$$

# Conclusion of the proof

- By considering a large control on a small time-interval we can approximately reach $e^{it(H_0+\frac{u_j}{t}H_j)} \xrightarrow[t\to 0]{} e^{iu_jH_j}$, for any $u_j \in \mathbb{R}$.

# Conclusion of the proof

- By considering a large control on a small time-interval we can approximately reach $e^{it(H_0+\frac{u_j}{t}H_j)} \xrightarrow[t\to 0]{} e^{iu_jH_j}$, for any $u_j \in \mathbb{R}$.
- By Poincaré Theorem, if the time is large enough, we can approximately reach $e^{isH_0}$, for any $s \in \mathbb{R}$.

# Conclusion of the proof

- By considering a large control on a small time-interval we can approximately reach $e^{it(H_0+\frac{u_j}{t}H_j)} \xrightarrow[t\to 0]{} e^{iu_jH_j}$, for any $u_j \in \mathbb{R}$.
- By Poincaré Theorem, if the time is large enough, we can approximately reach $e^{isH_0}$, for any $s \in \mathbb{R}$.
- If we can approximately reach $e^A\psi_0$ from any $\psi_0$, and $e^B\psi_0$ form any $\psi_0$, then we can approximately reach $e^Ae^B\psi_0$ form any $\psi_0$.

# Conclusion of the proof

- By considering a large control on a small time-interval we can approximately reach $e^{it(H_0+\frac{u_j}{t}H_j)} \xrightarrow[t\to 0]{} e^{iu_jH_j}$, for any $u_j \in \mathbb{R}$.
- By Poincaré Theorem, if the time is large enough, we can approximately reach $e^{isH_0}$, for any $s \in \mathbb{R}$.
- If we can approximately reach $e^A\psi_0$ from any $\psi_0$, and $e^B\psi_0$ form any $\psi_0$, then we can approximately reach $e^Ae^B\psi_0$ form any $\psi_0$.
- By Lemma 2 (and Lie-Trotter product formula), we can approximately reach $e^A$, for any $A \in \mathrm{Lie}\{iH_0, \dots, iH_m\}$.

# Conclusion of the proof

- By considering a large control on a small time-interval we can approximately reach $e^{it(H_0 + \frac{u_j}{t} H_j)} \xrightarrow[t \to 0]{} e^{iu_j H_j}$, for any $u_j \in \mathbb{R}$.
- By Poincaré Theorem, if the time is large enough, we can approximately reach $e^{isH_0}$, for any $s \in \mathbb{R}$.
- If we can approximately reach $e^A \psi_0$ from any $\psi_0$, and $e^B \psi_0$ form any $\psi_0$, then we can approximately reach $e^A e^B \psi_0$ form any $\psi_0$.
- By Lemma 2 (and Lie-Trotter product formula), we can approximately reach $e^A$, for any $A \in \mathrm{Lie}\{iH_0, \ldots, iH_m\}$.
- Since $\mathrm{Lie}\{iH_0, \ldots, iH_m\} = \mathfrak{u}(N)$, the surjectivity of $\exp : \mathfrak{u}(N) \to U(N)$ and the transitivity of $U(N) \circlearrowright \mathcal{S}^N$ imply that from any $\psi_0$ we can approximately reach any $\psi_1$.

# Small-time controllability

## The small-time controllability problem

Given $\psi_0, \psi_1 \in \mathcal{S}^n$ and $T > 0$, find $u : [0, \tau] \to R^m$ with $\tau \leqslant T$ such that $\psi(\tau, u, \psi_0) = \psi_1$.

# Small-time controllability

## The small-time controllability problem

Given $\psi_0, \psi_1 \in \mathcal{S}^n$ and $T > 0$, find $u : [0, \tau] \to R^m$ with $\tau \leqslant T$ such that $\psi(\tau, u, \psi_0) = \psi_1$.

## The small-time approximate controllability problem

Given $\psi_0, \psi_1 \in \mathcal{S}^n$, and an error $\varepsilon > 0$, find $u : [0, \tau] \to R^m$ with $\tau \leqslant \varepsilon$ such that $|\psi(\tau, u, \psi_0) - \psi_1| < \varepsilon$.

We say that the equation is small-time (approximately) controllable if the small-time (approximate) controllability problem is solvable for every $\psi_0, \psi_1 \in \mathcal{S}^n$.

# Small-time controllability

## The small-time controllability problem

Given $\psi_0, \psi_1 \in \mathcal{S}^n$ and $T > 0$, find $u : [0, \tau] \to R^m$ with $\tau \leqslant T$ such that $\psi(\tau, u, \psi_0) = \psi_1$.

## The small-time approximate controllability problem

Given $\psi_0, \psi_1 \in \mathcal{S}^n$, and an error $\varepsilon > 0$, find $u : [0, \tau] \to R^m$ with $\tau \leqslant \varepsilon$ such that $|\psi(\tau, u, \psi_0) - \psi_1| < \varepsilon$.

We say that the equation is small-time (approximately) controllable if the small-time (approximate) controllability problem is solvable for every $\psi_0, \psi_1 \in \mathcal{S}^n$.

## Min-time

Min-time$(\psi_0, \psi_1) = \inf\{t \geqslant 0 : \exists u : [0, t] \to \mathbb{R}^m \text{ s.t. } \psi(t, u, \psi_0) = \psi_1\}$.
The min-time of a system is the $\sup\{\text{min-time}(\psi_0, \psi_1), \psi_0, \psi_1 \in \mathcal{S}^N\}$.

Small-time controllability means that min-time$(\psi_0, \psi_1) = 0$ for any $\psi_0, \psi_1 \in \mathcal{S}^N$, or equivalently that its min-time is 0.

# A criterium for small-time controllability

## Theorem

The Schrödinger equation for the propagator is small-time controllable iff[3] $\mathrm{Lie}\{iH_1, \dots, iH_m\} = \mathfrak{u}(N)$. In particular, if $\mathrm{Lie}\{iH_1, \dots, iH_m\} = \mathfrak{u}(N)$ the Schrödinger equation for the state is small-time controllable.

[3] Agrachev, Boscain, Gauthier, Sigalotti; A note on time-zero controllability and density of orbits for quantum systems. Proceedings of IEEE 56th Annual Conference on Decision and Control (CDC) (2017)

[4] D'Alessandro; Small time controllability of systems on compact Lie groups and spin angular momentum. J. Math. Phys. 42, 4488 (2001)

# A criterium for small-time controllability

**Theorem**

The Schrödinger equation for the propagator is small-time controllable iff[3] $\mathrm{Lie}\{iH_1, \ldots, iH_m\} = \mathfrak{u}(N)$. In particular, if $\mathrm{Lie}\{iH_1, \ldots, iH_m\} = \mathfrak{u}(N)$ the Schrödinger equation for the state is small-time controllable.

Observe that $e^A$ is approximately reachable in small time for any $A \in \mathrm{Lie}\{iH_1, \ldots, iH_m\}$, by considering $e^{it(H_0 + \frac{u_j}{t} H_j)} \xrightarrow[t \to 0]{} e^{iu_j H_j}$. What takes long time is the recurrency of the drift, not used here.

[3] Agrachev, Boscain, Gauthier, Sigalotti; A note on time-zero controllability and density of orbits for quantum systems. Proceedings of IEEE 56th Annual Conference on Decision and Control (CDC) (2017)

[4] D'Alessandro; Small time controllability of systems on compact Lie groups and spin angular momentum. J. Math. Phys. 42, 4488 (2001)

# A criterium for small-time controllability

**Theorem**

The Schrödinger equation for the propagator is small-time controllable iff[3] $\mathrm{Lie}\{iH_1, \dots, iH_m\} = \mathfrak{u}(N)$. In particular, if $\mathrm{Lie}\{iH_1, \dots, iH_m\} = \mathfrak{u}(N)$ the Schrödinger equation for the state is small-time controllable.

Observe that $e^A$ is approximately reachable in small time for any $A \in \mathrm{Lie}\{iH_1, \dots, iH_m\}$, by considering $e^{it(H_0 + \frac{u_j}{t} H_j)} \xrightarrow[t \to 0]{} e^{iu_j H_j}$. What takes long time is the recurrency of the drift, not used here.

- The "if" part is true more in general[4] for any compact connected Lie group $G$, which in this case is $U(n)$. The "only if" in general is an open problem. If $G$ is not compact, the "only if" is false: $SL(2)$.

---

[3] Agrachev, Boscain, Gauthier, Sigalotti; A note on time-zero controllability and density of orbits for quantum systems. Proceedings of IEEE 56th Annual Conference on Decision and Control (CDC) (2017)

[4] D'Alessandro; Small time controllability of systems on compact Lie groups and spin angular momentum. J. Math. Phys. 42, 4488 (2001)

# A criterium for small-time controllability

**Theorem**

The Schrödinger equation for the propagator is small-time controllable iff[3] $\mathrm{Lie}\{iH_1, \dots, iH_m\} = \mathfrak{u}(N)$. In particular, if $\mathrm{Lie}\{iH_1, \dots, iH_m\} = \mathfrak{u}(N)$ the Schrödinger equation for the state is small-time controllable.

Observe that $e^A$ is approximately reachable in small time for any $A \in \mathrm{Lie}\{iH_1, \dots, iH_m\}$, by considering $e^{it(H_0 + \frac{u_j}{t} H_j)} \xrightarrow[t \to 0]{} e^{iu_j H_j}$. What takes long time is the recurrency of the drift, not used here.

- The "if" part is true more in general[4] for any compact connected Lie group $G$, which in this case is $U(n)$. The "only if" in general is an open problem. If $G$ is not compact, the "only if" is false: $SL(2)$.
- The converse (for the state) is also true when $N$ is odd (with $\mathfrak{su}(N)$ instead of $\mathfrak{u}(N)$), but not when $N$ is even.

[3] Agrachev, Boscain, Gauthier, Sigalotti; A note on time-zero controllability and density of orbits for quantum systems. Proceedings of IEEE 56th Annual Conference on Decision and Control (CDC) (2017)

[4] D'Alessandro; Small time controllability of systems on compact Lie groups and spin angular momentum. J. Math. Phys. 42, 4488 (2001)

# Example (1/2-spin)

Thanks to the commutation relations satisfied by the Pauli matrices, and by the criterium for controllability, the equation

$$i\frac{d}{dt}\psi(t) = (\sigma_z + u(t)\sigma_y)\psi(t),$$

with $u(t) \in \mathbb{R}, \psi \in \mathbb{C}^2, \|\psi\| = 1$, is controllable.

[5] Khaneja, Brockett, Glaser; Time Optimal Control in Spin Systems. Physical review A, 63(3), (2000)

[6] Agrachev, Chambrion: An estimation of the controllability time for single-input systems on compact Lie Groups, ESAIM COCV (2006)

[7] Rossi, Gauthier, A universal gap for non-spin quantum control systems, Proc. AMS (2021)

# Example (1/2-spin)

Thanks to the commutation relations satisfied by the Pauli matrices, and by the criterium for controllability, the equation

$$i\frac{d}{dt}\psi(t) = (\sigma_z + u(t)\sigma_y)\psi(t),$$

with $u(t) \in \mathbb{R}, \psi \in \mathbb{C}^2, \|\psi\| = 1$, is controllable.The criterium for STC is not satisfied. In fact, it is not STC.

[5] Khaneja, Brockett, Glaser; Time Optimal Control in Spin Systems. Physical review A, 63(3), (2000)

[6] Agrachev, Chambrion: An estimation of the controllability time for single-input systems on compact Lie Groups, ESAIM COCV (2006)

[7] Rossi, Gauthier, A universal gap for non-spin quantum control systems, Proc. AMS (2021)

# Example (1/2-spin)

Thanks to the commutation relations satisfied by the Pauli matrices, and by the criterium for controllability, the equation

$$i\frac{d}{dt}\psi(t) = (\sigma_z + u(t)\sigma_y)\psi(t),$$

with $u(t) \in \mathbb{R}, \psi \in \mathbb{C}^2, \|\psi\| = 1$, is controllable.The criterium for STC is not satisfied. In fact, it is not STC.Characterization of the min-time:

**Theorem**[5]

Given any $U \in U(2)$, there exists a unique $\tau \in [0, 2\pi)$ such that $U = U_1 e^{\tau\sigma_z} U_2$ where $U_2, U_1 \in e^{\mathbb{R}\sigma_x}$. Moreover, $\tau =$min-time$(I, U)$.

Estimating[6] the min-time is an (open) problem[7] with important technological consequences.

[5] Khaneja, Brockett, Glaser; Time Optimal Control in Spin Systems. Physical review A, 63(3), (2000)

[6] Agrachev, Chambrion: An estimation of the controllability time for single-input systems on compact Lie Groups, ESAIM COCV (2006)

[7] Rossi, Gauthier, A universal gap for non-spin quantum control systems, Proc. AMS (2021)

# Swapping the eigenstates of $H_1$ takes long time

We consider $\frac{d}{dt}\psi(t) = (A + uB)\psi(t), \psi \in \mathcal{S}^N$.

**Theorem[8]**

If $m = 1$ and $N \geqslant 2$, the equation for the state is not small-time controllable.

[8] Beauchard, Coron, Teismann, Minimal time for the bilinear control of Schrödinger equations, Sytems & Control Letters 71, 2014

# Swapping the eigenstates of $H_1$ takes long time

We consider $\frac{d}{dt}\psi(t) = (A + uB)\psi(t), \psi \in \mathcal{S}^N$.

**Theorem**[8]

If $m = 1$ and $N \geqslant 2$, the equation for the state is not small-time controllable.

Let $v(t) = \int_0^t u(s)ds$, $\phi(t) = e^{-v(t)}\psi(t)$, then

$$\frac{d}{dt}\phi(t) = e^{-v(t)B}Ae^{v(t)B}\phi(t).$$

[8] Beauchard, Coron, Teismann, Minimal time for the bilinear control of Schrödinger equations, Sytems & Control Letters 71, 2014

# Swapping the eigenstates of $H_1$ takes long time

We consider $\frac{d}{dt}\psi(t) = (A + uB)\psi(t), \psi \in \mathcal{S}^N$.

**Theorem**[8]

If $m = 1$ and $N \geqslant 2$, the equation for the state is not small-time controllable.

Let $v(t) = \int_0^t u(s)ds$, $\phi(t) = e^{-v(t)}\psi(t)$, then

$$\frac{d}{dt}\phi(t) = e^{-v(t)B} A e^{v(t)B} \phi(t).$$

Let $e_j$ eigenbasis for $B$: $Be_j = b_j e_j, b_j \in i\mathbb{R}$. Then $|\langle\psi(t), e_j\rangle| = |\langle\phi(t), e_j\rangle|$.

[8] Beauchard, Coron, Teismann, Minimal time for the bilinear control of Schrödinger equations, Sytems & Control Letters 71, 2014

# Swapping the eigenstates of $H_1$ takes long time

We consider $\frac{d}{dt}\psi(t) = (A + uB)\psi(t), \psi \in \mathcal{S}^N$.

**Theorem**[8]

If $m = 1$ and $N \geqslant 2$, the equation for the state is not small-time controllable.

Let $v(t) = \int_0^t u(s)ds$, $\phi(t) = e^{-v(t)}\psi(t)$, then

$$\frac{d}{dt}\phi(t) = e^{-v(t)B}Ae^{v(t)B}\phi(t).$$

Let $e_j$ eigenbasis for $B$: $Be_j = b_j e_j, b_j \in i\mathbb{R}$. Then $|\langle\psi(t), e_j\rangle| = |\langle\phi(t), e_j\rangle|$.Hence

$$|\langle\psi(t), e_j\rangle| - |\langle\psi_0, e_j\rangle| = |\langle\phi(t), e_j\rangle| - |\langle\phi_0, e_j\rangle| \leqslant |\langle\phi(t) - \phi_0, e_j\rangle|$$
$$= |\langle\int_0^t \phi'(s)ds, e_j\rangle| \leqslant t\|A\| \Rightarrow \text{min-time}(e_j, e_k) \geqslant 1/\|A\|.$$

[8] Beauchard, Coron, Teismann, Minimal time for the bilinear control of Schrödinger equations, Sytems & Control Letters 71, 2014

# Swapping the eigenstates of $H_1$ takes long time

We consider $\frac{d}{dt}\psi(t) = (A + uB)\psi(t), \psi \in \mathcal{S}^N$.

**Theorem[8]**

If $m = 1$ and $N \geqslant 2$, the equation for the state is not small-time controllable.

Let $v(t) = \int_0^t u(s)ds$, $\phi(t) = e^{-v(t)}\psi(t)$, then

$$\frac{d}{dt}\phi(t) = e^{-v(t)B} A e^{v(t)B} \phi(t).$$

Let $e_j$ eigenbasis for $B$: $Be_j = b_j e_j, b_j \in i\mathbb{R}$. Then $|\langle \psi(t), e_j \rangle| = |\langle \phi(t), e_j \rangle|$.Hence

$$\begin{aligned} |\langle \psi(t), e_j \rangle| - |\langle \psi_0, e_j \rangle| &= |\langle \phi(t), e_j \rangle| - |\langle \phi_0, e_j \rangle| \leqslant |\langle \phi(t) - \phi_0, e_j \rangle| \\ &= |\langle \int_0^t \phi'(s)ds, e_j \rangle| \leqslant t\|A\| \Rightarrow \text{min-time}(e_j, e_k) \geqslant 1/\|A\|. \end{aligned}$$

What happens if $A$ is unbounded?

[8] Beauchard, Coron, Teismann, Minimal time for the bilinear control of Schrödinger equations, Sytems & Control Letters 71, 2014

# An example of a STC scalar-input system

The group of real $2 \times 2$ matrices with det=1 is $SL(2)$. Its Lie algebra is given by the $2 \times 2$ real traceless matrices $\mathfrak{sl}(2)$, with basis:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutation relations: $[a, b] = c, [a, c] = 2a, [b, c] = -2b.$

**Theorem**[9]

$\dot{g} = (a + ub)g, g \in SL(2)$, is STC.

[9] Beauchard, Pozzoli: Examples of small-time controllable Schrödinger equations. Annales Henri Poincaré (2025)

# An example of a STC scalar-input system

The group of real $2 \times 2$ matrices with det=1 is $SL(2)$. Its Lie algebra is given by the $2 \times 2$ real traceless matrices $\mathfrak{sl}(2)$, with basis:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutation relations: $[a, b] = c, [a, c] = 2a, [b, c] = -2b.$

## Theorem[9]

$\dot{g} = (a + ub)g, g \in SL(2)$, is STC.

Ingredients:

- $e^{-u\frac{b}{\tau}} e^{\tau(a - \frac{2u^2 b}{\tau^2})} e^{u\frac{b}{\tau}} \longrightarrow e^{ub}$ as $\tau \to 0$, for any $u \in \mathbb{R}$.

[9] Beauchard, Pozzoli: Examples of small-time controllable Schrödinger equations. Annales Henri Poincaré (2025)

# An example of a STC scalar-input system

The group of real $2 \times 2$ matrices with det=1 is $SL(2)$. Its Lie algebra is given by the $2 \times 2$ real traceless matrices $\mathfrak{sl}(2)$, with basis:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutation relations: $[a, b] = c, [a, c] = 2a, [b, c] = -2b$.

**Theorem**[9]

$\dot{g} = (a + ub)g, g \in SL(2)$, is STC.

Ingredients:

- $e^{-u\frac{b}{\tau}} e^{\tau(a - \frac{2u^2 b}{\tau^2})} e^{u\frac{b}{\tau}} \longrightarrow e^{ub}$ as $\tau \to 0$, for any $u \in \mathbb{R}$.
- $e^{\log(\tau^{1/2})c} e^{s\tau a} e^{\log(\tau^{-1/2}c)} = e^{sa}$, for any $s, \tau > 0$.

[9] Beauchard, Pozzoli: Examples of small-time controllable Schrödinger equations. Annales Henri Poincaré (2025)

# An example of a STC scalar-input system

The group of real $2 \times 2$ matrices with det=1 is $SL(2)$. Its Lie algebra is given by the $2 \times 2$ real traceless matrices $\mathfrak{sl}(2)$, with basis:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutation relations: $[a, b] = c, [a, c] = 2a, [b, c] = -2b.$

## Theorem[9]

$\dot{g} = (a + ub)g, g \in SL(2)$, is STC.

Ingredients:

- $e^{-u\frac{b}{\tau}} e^{\tau(a - \frac{2u^2 b}{\tau^2})} e^{u\frac{b}{\tau}} \longrightarrow e^{ub}$ as $\tau \to 0$, for any $u \in \mathbb{R}$.
- $e^{\log(\tau^{1/2})c} e^{s\tau a} e^{\log(\tau^{-1/2}c)} = e^{sa}$, for any $s, \tau > 0$.
- $e^{s(a-b)} = \begin{pmatrix} \sin(s) & \cos(s) \\ -\cos(s) & \sin(s) \end{pmatrix}$ is periodic (thus recurrent!) in $s$.

  Hence I can generate $e^{s(a-b)}, s < 0$.

[9]Beauchard, Pozzoli: Examples of small-time controllable Schrödinger equations. Annales Henri Poincaré (2025)

# An example of a STC scalar-input system

The group of real $2 \times 2$ matrices with det=1 is $SL(2)$. Its Lie algebra is given by the $2 \times 2$ real traceless matrices $\mathfrak{sl}(2)$, with basis:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Commutation relations: $[a, b] = c, [a, c] = 2a, [b, c] = -2b$.

**Theorem**[9]

$\dot{g} = (a + ub)g, g \in SL(2)$, is STC.

Ingredients:

- $e^{-u\frac{b}{\tau}} e^{\tau(a - \frac{2u^2}{\tau^2}b)} e^{u\frac{b}{\tau}} \longrightarrow e^{ub}$ as $\tau \to 0$, for any $u \in \mathbb{R}$.
- $e^{\log(\tau^{1/2})c} e^{s\tau a} e^{\log(\tau^{-1/2}c)} = e^{sa}$, for any $s, \tau > 0$.
- $e^{s(a-b)} = \begin{pmatrix} \sin(s) & \cos(s) \\ -\cos(s) & \sin(s) \end{pmatrix}$ is periodic (thus recurrent!) in $s$.

  Hence I can generate $e^{s(a-b)}, s < 0$.

Weyl metaplectic representation, harmonic oscillator: $a = i\Delta, b = i\|x\|^2$.

[9]Beauchard, Pozzoli: Examples of small-time controllable Schrödinger equations. Annales Henri Poincaré (2025)

To be continued...
Thanks for your attention!