Controllability of Schrödinger equations and application to quantum rotors

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1: Controllability of finite-dimensional Schrödinger equations
2: Spectral conditions and resonant control
3: Controllability of ∞-dimensional Schrödinger equations

Controllability of ∞-dimensional Schrödinger equations

Quantum rotors and BEC on $\mathbb{T}^d$

On $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$,

$$i\partial_t\psi(t,x) = \Big(-\Delta + V(x) + \sum_{j=1}^{d}(u_{2j-1}(t)\sin + u_{2j}(t)\cos)\langle b_j, x\rangle\Big)\psi(t,x) + \kappa|\psi|^{2p}\psi,$$

where $p \in \mathbb{N}, \kappa \in \mathbb{R},\ V \in L^\infty(\mathbb{T}^d, \mathbb{R})$, and

$$b_1 = (1,0,\dots,0), \quad \dots, \quad b_{d-1} = (0,\dots,1,0), \quad b_d = (1,\dots,1).$$

It is locally-in-time well-posed[1]. We shall study small-time controllability, which is thus a well-defined notion if the time is small enough such that the equation is locally well-posed.

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In $d = 1$, it is just a rewriting of

$$i\partial_t\psi(t,x) = \Big(- \Delta + V(x) + s(t)\cos(x + \varphi(t))\Big)\psi(t,x) + \kappa|\psi|^{2p}\psi,$$

with $u_1 = s\cos(\varphi), u_2 = -s\sin(\varphi)$.

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Notion of small-time approximate controllability

Introduce the L^2-unitary sphere

$$\mathcal{S} := \{\psi \in L^2(M, \mathbb{C})\,;\, \|\psi\|_{L^2(M)} = 1\}.$$

Denote by $\psi(t; u, \psi_0)$ the solution: $\psi_0 \in \mathcal{S} \Rightarrow \psi(t; u, \psi_0) \in \mathcal{S}$, for all $t \in \mathbb{R}$ such that the equation is well-defined, and for all $u \in PWC$.

Definition 1 (STAC)

The equation is **small-time approximately controllable** (STAC) if, for every $\psi_0, \psi_1 \in \mathcal{S}$ and $\varepsilon > 0$, there exist a time $T \in [0, \varepsilon]$, a global phase $\theta \in [0, 2\pi)$ and a PWC control $u : [0, T] \to \mathbb{R}^m$ such that

$$\|\psi(T; u, \psi_0) - e^{i\theta}\psi_1\|_{L^2} < \varepsilon.$$

A recent controllability result

We shall prove the following result for simplicity in the linear case $\kappa = 0$, but it holds also in the nonlinear case[2].

Theorem[3]

The Schrödinger equation on $\mathbb{T}^d$ is small-time approximately controllable.

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- The same methods apply also to equations with purely continuous spectrum, e.g.

$$i\partial_t\psi(x,t) = (-\Delta + u(t)e^{-x^2/2} + v(t)x)\psi(x,t), \quad x \in \mathbb{R}.$$

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- An open problem is to find a ***scalar-input*** Schrödinger equation on $L^2(M)$ which is STAC[4].

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Some notions I

Definition 2 (STAR operators)

A unitary operator L on $L^2(M,\mathbb{C})$ is STAR if, for every $\psi_0 \in \mathcal{S}$ and $\epsilon > 0$, there exist $T \in [0,\epsilon]$, $\theta \in [0,2\pi)$ and $u \in PWC(0,T)$ such that $\|\psi(T;u,\psi_0) - e^{i\theta}L\psi_0\|_{L^2} < \epsilon$.

Lemma 3

The composition and the strong limit of STAR operators are STAR operators.

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Definition 4 (Unitary action of $\mathrm{Diff}^0(M)$ on $L^2(M, \mathbb{C})$)

For $P \in \mathrm{Diff}^0(M)$, the unitary operator on $L^2(M, \mathbb{C})$ associated with P is defined by

$$\mathcal{L}_P \psi = |J_P|^{1/2}(\psi \circ P),$$

where $|J_P| = \det(DP)$ is the determinant of the Jacobian matrix of P. Then $\|\mathcal{L}_P \psi\|_{L^2} = \|\psi\|_{L^2}$.

Some notions II

Definition 5 (Flows ϕ_f^s)

For $f \in \mathrm{Vec}(M)$, ϕ_f^s denotes the flow associated with f at time s: for every $x_0 \in M$, $x(s) = \phi_f^s(x_0)$ is the solution of the ODE

$$\dot{x}(s) = f(x(s)), \quad x(0) = x_0.$$

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Definition 6 (STC notions)

- **STC of phases**: for every $\varphi \in L^2(M, \mathbb{R})$, the operator $e^{i\varphi}$ is STAR,
- **STC of the group** $\mathrm{Diff}_c^0(M)$: for every $P \in \mathrm{Diff}_c^0(M)$, the operator $\mathcal{L}_P$ is STAR.
- **STC of flows**: for every $f \in \mathrm{Vec}_c(M)$, the operator $\mathcal{L}_{\phi_f^1}$ is STAR.

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- **STC of flows**: for every $f \in \mathrm{Vec}_c(M)$, the operator $\mathcal{L}_{\phi_f^1}$ is STAR.

For $f \in \mathrm{Vec}(M)$ (resp. $\mathrm{Vec}_c(M)$) then $\phi_f^1 \in \mathrm{Diff}^0(M)$ (resp. $\mathrm{Diff}_c^0(M)$). Hence the following implication trivially holds

$$\text{STC of the group } \mathrm{Diff}_c^0(M) \qquad \Rightarrow \qquad \text{STC of flows.}$$

A Theorem of Moser[5]

Transitivity of the group action of $\mathrm{Diff}(M)$ on positive probability densities.

Moser's Theorem

Let M be compact. Given $\rho_0, \rho_1 \in C^\infty(M, (0, \infty))$, there exists $P \in \mathrm{Diff}^0(\mathrm{M})$ s.t.

$$|J_P(x)|^{1/2} \rho_0(P(x)) = \rho_1(x)$$

iff

$$\int_M \rho_0^2 dx = \int_M \rho_1^2 dx.$$

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E.g. on $\mathbb{T}$: let $\|\rho_0\|_{L^2} = \|\rho_1\|_{L^2} = C$, and let $\rho \equiv (2\pi)^{-1/2} C^{1/2}$.

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$$\mathcal{L}_{P_j} \rho = \rho_j, \quad j = 0, 1$$

then $P = P_2 P_1^{-1}$ is such that $\mathcal{L}_P \rho_0 = \rho_1$.

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then $P = P_2 P_1^{-1}$ is such that $\mathcal{L}_P \rho_0 = \rho_1$.The P_j may be taken to be

$$P_j(x) = 2\pi C^{-1}\int_0^x \rho_j^2(y)dy, \quad j = 0, 1.$$

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Reduction to the control of phases and diffeos

The first step consists in proving a certain ***transitivity*** property.

Theorem

$$\textit{STC of phases and the group } \mathrm{Diff}^0_c(M) \qquad \Rightarrow \qquad \textit{STAC.}$$

This is a consequence of Moser's Theorem, applied to the radial part of the wavefunction's ***polar decomposition***.

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By density, and since the solution operator is an isometry, WLOG

$$\psi_j = \rho_j e^{i\phi_j}, \quad \phi_j \in L^2(M, \mathbb{R}), \quad \rho_j \in C^\infty(M, (0, \infty)).$$

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Moser's Theorem implies the existence of $P \in \mathrm{Diff}^0_c(M)$ such that

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Then $e^{i\phi_1} \mathcal{L}_P e^{-i\phi_0} \psi_0 = e^{i\phi_1} \mathcal{L}_P \rho_0 = e^{i\phi_1} \rho_1 = \psi_1$. By Lemma 3, the operator $e^{i\phi_1} \mathcal{L}_P e^{-i\phi_0}$ is STAR.

A Theorem of Thurston[6]

Definition of simple group

A subgroup H of a group G is said to be *normal* if, for any $h \in H$ and $g \in G$, $ghg^{-1} \in H$. A group G is said to be *simple* if its only normal subgroups are G and $\{id\}$.

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This gives a certain ***surjectivity*** property.

Consequence: decomposition of diffeo as product of flows

Given $P \in \mathrm{Diff}_c^0(M)$ there exist $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \mathrm{Vec}_c(M)$ such that

$$P = \phi_{f_n}^1 \circ \cdots \circ \phi_{f_1}^1.$$

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Proof of the consequence

The subgroup

$$F(M) := \{\phi^1_{f_n} \circ \cdots \circ \phi^1_{f_1};\, n \in \mathbb{N}^*, f_1, \ldots, f_n \in \mathrm{Vec}_c(M)\}$$

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$$g_j(x) := (P \star f_j)(x) = DP(P^{-1}(x))\, f_j(P^{-1}(x)),$$

thus $PXP^{-1} \in F(M)$.

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$$g_j(x) := (P \star f_j)(x) = DP(P^{-1}(x))\, f_j(P^{-1}(x)),$$

thus $PXP^{-1} \in F(M)$.By Thurston's Theorem, $\mathrm{Diff}^0_c(M) = F(M)$.

Reduction to the control of phases and flows

The second step consists in proving

Theorem

STC of flows $\Rightarrow$ *STC of the group* $\mathrm{Diff}_c^0(M)$.

This is a simple application of the consequence of Thurston's Theorem.

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This is a simple application of the consequence of Thurston's Theorem.We thus get the following criterium.

Corollary

STC of phases and flows $\Rightarrow$ *STAC.*

We are left to show the STAC of phases and flows for the Schrödinger equation on $\mathbb{T}^d$.

STC of phases I

Theorem[7]

The STC of phases holds for the Schrödinger equation on $\mathbb{T}^d$.

The main ingredients are:

- $$e^{-i\tau^{-1/2}W(x)}e^{i\tau(\Delta-V(x))}e^{i\tau^{-1/2}W(x)}\psi_0 \xrightarrow{\tau\to 0} e^{-i|\nabla W|^2}\psi_0.$$

  Which follows from

  $$e^{-i\tau^{-1/2}W(x)}e^{i\tau(\Delta-V(x))}e^{i\tau^{-1/2}W(x)}\psi_0 = \exp\left(i\tau(\Delta-V)+\tau^{1/2}\mathcal{T}_f - i|\nabla W|^2\right)$$

  where $f=\nabla W$, $\mathcal{T}_f=\langle f,\nabla\cdot\rangle+\frac{1}{2}\mathrm{div}(f)$. It is an explicit sequence of delta kicks.

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  where $f = \nabla W$, $\mathcal{T}_f = \langle f, \nabla \cdot \rangle + \frac{1}{2}\mathrm{div}(f)$. It is an explicit sequence of delta kicks.

- And the density of a certain ***infinite-dimensional Lie algebra***.

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STC of phases II

We next introduce $\mathcal{H}_0 = \mathrm{span}\{V_1, \dots, V_{2d+1}\}$ and $\mathcal{H}_j$ the largest vector space whose elements can be written as

$$\phi_0 - \sum_{i=1}^{N} |\nabla \phi_i|^2, \quad \phi_i \in \mathcal{H}_{j-1},\ N \in \mathbb{N}.$$

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Then, we have that $\cup_{j=0}^{\infty} \mathcal{H}_j$ is ***dense*** in $L^2(\mathbb{T}^d, \mathbb{R})$. The idea is that

$$2|\cos'(x)|^2 = 2\sin^2(x) = 1 - \cos(2x) \cong -\cos(2x),$$

and

$$2|\sin'(x)|^2 = 2\cos^2(x) = 1 + \cos(2x) \cong +\cos(2x).$$

Strategy to show STC of flows

We next show the following.

Theorem

The STC of flows holds for the Schrödinger equations on $\mathbb{T}^d$.

In view of the previous slides, the latter Theorem implies the main theorem, i.e. the STAC of the Schrödinger equations on $\mathbb{T}^d$.

Strategy to show STC of flows

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In view of the previous slides, the latter Theorem implies the main theorem, i.e. the STAC of the Schrödinger equations on $\mathbb{T}^d$.
Our strategy to prove it is:

STC of phases $\Rightarrow$ STC of flows of gradient vector fields $\Rightarrow$ STC of flows.

STC of phases $\Rightarrow$ STC of flows of gradient vector fields I

Introduce the space of gradient vector fields on $\mathbb{T}^d$

$$\mathfrak{G} = \{\nabla\varphi; \varphi \in C^\infty(\mathbb{T}^d, \mathbb{R})\} \subset \mathrm{Vec}(\mathbb{T}^d).$$

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Theorem

Let $f = \nabla\varphi \in \mathfrak{G}$ and $P := \phi_f^1$. Then $\mathcal{L}_P$ is STAR for the Schrödinger equations on $\mathbb{T}^d$ and $\mathbb{R}^d$.

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Theorem

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The first remark is that, being $\mathcal{L}_P$ unitary, it must be the exponential of some skew-adjoint operator (***Stone Theorem***). We have

$$(\mathcal{L}_{\phi_f^t}\psi)(x) = \psi(\phi_f^t(x))e^{\frac{1}{2}\int_0^t \mathrm{div} f(\phi_f^s(x))ds} = (e^{t\mathcal{T}_f}\psi)(x)$$

where $\mathcal{T}_f = \langle f, \nabla\cdot\rangle + \frac{1}{2}\mathrm{div}(f)$ is a skew-adjoint operator. In the first equality we used Liouville formula, and in the second equality the method of characteristics.

STC of phases ⇒ STC of flows of gradient vector fields II

The proof then consists in showing the following strong convergences:

$$\Bigg(e^{\frac{1}{n}\frac{i|\nabla\varphi|^2}{4\tau}} \underbrace{e^{i\frac{\varphi}{2\tau}} e^{i\frac{\tau}{n}(\Delta - V)} e^{-i\frac{\varphi}{2\tau}}}_{\exp\frac{1}{n}\left(i\tau(\Delta-V)+\mathcal{T}_f-\frac{i|\nabla\varphi|^2}{4\tau}\right)} \Bigg)^n \underset{n\to\infty}{\longrightarrow} e^{i\tau(\Delta-V)+\mathcal{T}_f} \underset{\tau\to 0}{\longrightarrow} e^{\mathcal{T}_f} = \mathcal{L}_P$$

where $\mathcal{T}_f = \langle f, \nabla\cdot\rangle + \frac{1}{2}\mathrm{div}(f)$. In the first convergence we used Trotter-Kato product formula $\lim_{n\to\infty}(e^{\frac{A}{n}}e^{\frac{B}{n}})^n = e^{A+B}$.

Theorem

$\mathfrak{L} := \{f \in \mathrm{Vec}(M); \forall t \in \mathbb{R}, \mathcal{L}_{\phi_f^t}$ is L^2-STAR$\}$ is a Lie subalgebra of $\mathrm{Vec}(M)$. Hence, by the previous Theorem, Lie$(\mathfrak{G}) \subset \mathfrak{L}$.

We show that if $f, g \in \mathfrak{L}$ and $P := \phi^1_{[f,g]}$ then $\mathcal{L}_P$ is the strong limit of STAR operators.

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We show that if $f, g \in \mathfrak{L}$ and $P := \phi^1_{[f,g]}$ then $\mathcal{L}_P$ is the strong limit of STAR operators.We use the following convergences

$$\left(e^{\frac{-1}{tn}\mathcal{T}_f} e^{-t\mathcal{T}_g} e^{\frac{1}{tn}\mathcal{T}_f} e^{t\mathcal{T}_g}\right)^n \underset{n\to\infty}{\longrightarrow} \exp\left(\frac{-1}{t}\mathcal{T}_f + e^{-t\mathcal{T}_g}\frac{1}{t}\mathcal{T}_f e^{t\mathcal{T}_g}\right) \underset{t\to 0}{\longrightarrow} e^{\mathcal{T}_{[f,g]}} = \mathcal{L}_P.$$

Do you remember the finite-dimensional analogue?

The last step of our strategy is proving that $\mathrm{Lie}(\mathfrak{G})$ is dense in $\mathrm{Vec}(\mathbb{T}^d)$.

Theorem

For every $f \in \mathrm{Vec}(\mathbb{T}^d)$, there exists $(f_n)_{n\in\mathbb{N}} \subset \mathrm{Lie}(\mathfrak{G})$ such that $\mathcal{L}_{\phi_f^1}$ is the strong limit of $(\mathcal{L}_{\phi_{f_n}^1})_{n\in\mathbb{N}}$.

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The basic idea is that the Lie bracket of gradient vector fields is not a gradient vector field. E.g.,

$$[\nabla \cos(x_j), \nabla \sin(x_j)] = [\cos(x_j)\partial_{x_j}, \sin(x_j)\partial_{x_j}] = \partial_{x_j},$$

and

$$-\frac{1}{2}[\nabla \sin(x_j)\sin(x_k), \nabla \cos(x_k)] + \frac{1}{2}[\nabla \sin(x_j)\cos(x_k), \nabla \sin(x_k)] = \sin(x_j)\partial_{x_k}.$$

Thanks for your attention!