Derivation of the Effective Dynamics for the Bose Polaron

Université de Bourgogne
Winter School: Physics and Mathematics of Bose-Einstein Condensates

SIEGFRIED SPRUCK

Jonas Lampart

Peter Pickl

February 24, 2025

Université de Bourgogne
Eberhard Karls Universität Tübingen

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

UNIVERSITÉ DE BOURGOGNE

CRC TRR 352

Plan

1. Motivation: Impurity Particles in Experiments
2. System Set-Up: Bose Gas in Condensation
3. Methods: Impurity Localization
4. Main Theorem: Validity Bose Polaron Dynamics

Motivation: Impurity Particle

Quantum gas of N bosons with 1 impurity particle $\rightarrow$ Tracer particle.

Applications of **impurity** systems in physics:

- **Track local structure**:
  Vortex lattice in liquid Helium.
- **Probing** the **density distribution** of a Bose gas [Schmid-Härter-Denschlag 10].

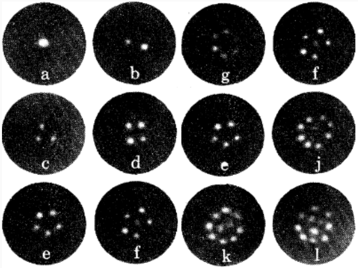


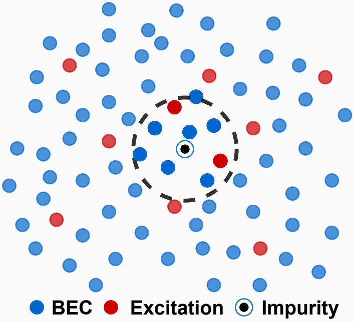
Figure 1: Rotating Helium with marked vortices by tracer particles [Varmchuk-Gordon-Packard 79].

Motivation: Impurity Particle

Quantum gas of N bosons with 1 impurity particle $\rightarrow$ Tracer particle.

Applications of **impurity** systems in physics:

- **Track local structure**:
  Vortex lattice in liquid Helium.
- **Probing** the **density distribution** of a Bose gas [Schmid-Härter-Denschlag 10].



Bose Polaron: Quasi-particle of impurity and bosons.

Goal: Prove the existence of a Bose Polaron in our system.
$\rightarrow$ Effective dynamics.

Next Step

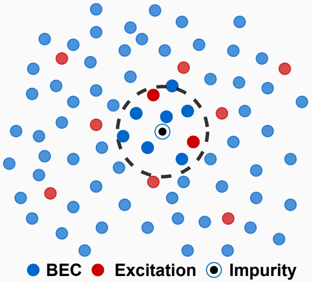
1. Motivation: Impurity Particles in Experiments
2. System Set-Up: Bose Gas in Condensation
3. Methods: Impurity Localization
4. Main Theorem: Validity Bose Polaron Dynamics

System Set-Up

- Quantum gas of N bosons, 1 impurity particle in $\mathbb{R}^3$.
- Initial **volume** Λ, **density** $\rho = \frac{N}{\Lambda}$ with ρ, Λ **large**.
- **Interactions** $V, W \in \mathscr{S}(\mathbb{R}^3, \mathbb{R})$ **weak**, even and of range $\mathcal{O}(1)$: **Mean-field scaling**.
- Dynamics on $L^2(\mathbb{R}^3_x) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3N}_y)$

$$
\begin{aligned}
i\partial_t \psi_{N,t} &= H_N \psi_{N,t}\,, \\
H_N &= -\sum_{i=1}^{N} \frac{\Delta_{y_i}}{2} - \frac{\Delta_x}{2m} + \frac{1}{\rho} \sum_{1 \le i < j \le N} V(y_i - y_j) \\
&\quad + \frac{1}{\sqrt{\rho}} \sum_{i=1}^{N} W(x - y_i)\,.
\end{aligned}
$$

x: Impurity position; y_i: Boson positions.



Bose-Einstein Condensate

Complete **Bose-Einstein condensation**
if almost all bosons in same state:

$$\psi_{N,t}(y_1, \ldots, y_N) \sim \prod_{i=1}^{N} \varphi_t(y_i)\,,$$

for large ρ, $\varphi_t \in L^2(\mathbb{R}^3)$ a **one-particle state**.

Bose-Einstein Condensate

Complete **Bose-Einstein condensation**
if almost all bosons in same state:

$$\psi_{N,t}(y_1, \dots, y_N) \sim \prod_{i=1}^{N} \varphi_t(y_i)\,,$$

for large ρ, $\varphi_t \in L^2(\mathbb{R}^3)$ a **one-particle state**.

Time evolution of **condensate** φ_t:

$$i\partial_t \varphi_t(y) = \Big(-\frac{\Delta}{2} + V \star |\varphi_t|^2(y) - \underbrace{\mu_t}_{\in\mathbb{R}} \Big)\varphi_t(y) \quad \textbf{(Hartree-eq)}\,,$$

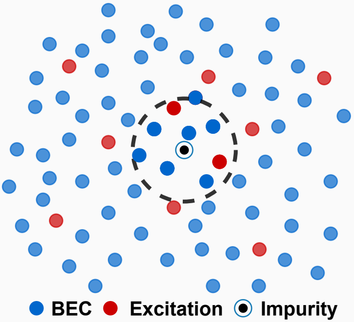
$$\varphi_{t=0} = \varphi_0\,.$$

Bose-Einstein Condensate

Complete **Bose-Einstein condensation** if almost all bosons in same state:

$$\psi_{N,t}(y_1, \ldots, y_N) \sim \prod_{i=1}^{N} \varphi_t(y_i)\,,$$

for large ρ, $\varphi_t \in L^2(\mathbb{R}^3)$ a **one-particle state**.



Time evolution of **condensate** φ_t:

$$i\partial_t \varphi_t(y) = \Big(-\frac{\Delta}{2} + V \star |\varphi_t|^2(y) - \underbrace{\mu_t}_{\in \mathbb{R}} \Big) \varphi_t(y) \quad \textbf{(Hartree-eq)}\,,$$

$$\varphi_{t=0} = \varphi_0\,.$$

The **condensate** φ_t defines a **background/environment** for the **excitation dynamics** we are actually interest in.

Excitations

We call $\zeta \in \{\varphi_t\}^\perp \subset L^2(\mathbb{R}^3) = \mathrm{lin}\{\varphi_t\} \oplus \{\varphi_t\}^\perp$ an **excitation** out of the condensate. These excitations can emerge and disappear. Define U_t, **isometry**, mapping into the **excitation space** $\mathcal{F}(\{\varphi_t\}^\perp)$

$$U_t : L^2_{\mathrm{sym}}(\mathbb{R}^{3N}) \to \mathcal{F}(\{\varphi_t\}^\perp) = \bigoplus_{k=0}^{\infty} (\{\varphi_t\}^\perp)^{\otimes_s k}$$

$$\underbrace{\varphi_t \otimes_s \cdots \otimes_s \varphi_t}_{(N-k)\text{ times}} \otimes_s \underbrace{\zeta_1 \otimes_s \ldots \zeta_k}_{\in (\{\varphi_t\}^\perp)^{\otimes_s k}} \stackrel{U_t}{\longmapsto} \zeta_1 \otimes_s \ldots \zeta_k \,.$$

Excitations

We call $\zeta \in \{\varphi_t\}^\perp \subset L^2(\mathbb{R}^3) = \mathrm{lin}\{\varphi_t\} \oplus \{\varphi_t\}^\perp$ an **excitation** out of the condensate. These excitations can emerge and disappear. Define U_t, **isometry**, mapping into the **excitation space** $\mathcal{F}(\{\varphi_t\}^\perp)$

$$U_t : L^2_{\mathrm{sym}}(\mathbb{R}^{3N}) \to \mathcal{F}(\{\varphi_t\}^\perp) = \bigoplus_{k=0}^{\infty} (\{\varphi_t\}^\perp)^{\otimes_s k}$$

$$\underbrace{\varphi_t \otimes_s \cdots \otimes_s \varphi_t}_{(N-k)\text{ times}} \otimes_s \underbrace{\zeta_1 \otimes_s \ldots \zeta_k}_{\in (\{\varphi_t\}^\perp)^{\otimes_s k}} \overset{U_t}{\longmapsto} \zeta_1 \otimes_s \ldots \zeta_k \,.$$

$U_t \psi_{N,t}$: excitation part of the wave function evolved by

$$i\partial_t U_t \psi_{N,t} = H^{\mathrm{ex}} U_t \psi_{N,t}\,,$$

$$H^{\mathrm{ex}} = U_t H_N U_t^* + i(\partial_t U_t) U_t^* \quad \textbf{(Excitation Hamiltonian)}\,.$$

Excitations

We call $\zeta \in \{\varphi_t\}^\perp \subset L^2(\mathbb{R}^3) = \mathrm{lin}\{\varphi_t\} \oplus \{\varphi_t\}^\perp$ an **excitation** out of the condensate. These excitations can emerge and disappear. Define U_t, **isometry**, mapping into the **excitation space** $\mathcal{F}(\{\varphi_t\}^\perp)$

$$U_t : L^2_{\mathrm{sym}}(\mathbb{R}^{3N}) \to \mathcal{F}(\{\varphi_t\}^\perp) = \bigoplus_{k=0}^{\infty} (\{\varphi_t\}^\perp)^{\otimes_s k}$$

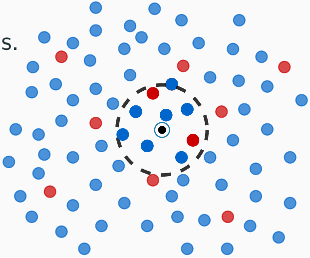
$$\underbrace{\varphi_t \otimes_s \cdots \otimes_s \varphi_t}_{(N-k)\text{ times}} \otimes_s \underbrace{\zeta_1 \otimes_s \ldots \zeta_k}_{\in (\{\varphi_t\}^\perp)^{\otimes_s k}} \xmapsto{U_t} \zeta_1 \otimes_s \ldots \zeta_k \,.$$

$U_t \psi_{N,t}$: excitation part of the wave function evolved by

$$i\partial_t U_t \psi_{N,t} = H^{\mathrm{ex}} U_t \psi_{N,t}\,,$$
$$H^{\mathrm{ex}} = U_t H_N U_t^* + i(\partial_t U_t) U_t^* \quad \textbf{(Excitation Hamiltonian)}\,.$$

Our setting: System exhibits **Bose-Einstein condensation** with **few excitations**.

Bose Polaron



Bose Polaron: Quasi-particle of impurity and bosons.
It is described by effective dynamics generated by **Bogoliubov-Fröhlich** Hamiltonian H^{BF}

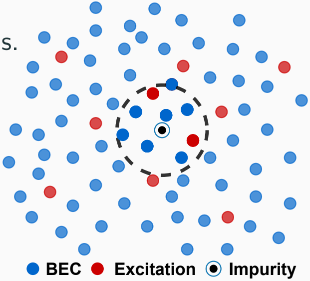
$$H^{\mathrm{BF}} = H^{\mathrm{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)\,,$$

$$i\partial_t \psi_t^{\mathrm{BF}} = H^{\mathrm{BF}} \psi_t^{\mathrm{BF}}\,,$$

on $L^2(\mathbb{R}^3_x) \otimes \mathcal{F}(\{\varphi_t\}^\perp)$.

$H^{\mathrm{Bog}} \sim d\Gamma(\xi)$ Bogoliubov Hamiltonian, modelling free excitations. Q_t projects into $\{\varphi_t\}^\perp$. $a^\#(Q_t W_x \varphi_t)$ **creates or annihilates excitation** due to interaction of impurity with condensate.

Bose Polaron



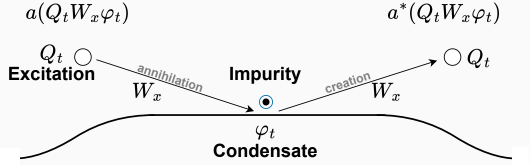
Bose Polaron: Quasi-particle of impurity and bosons.
It is described by effective dynamics generated by **Bogoliubov-Fröhlich** Hamiltonian H^{BF}

$$H^{\mathrm{BF}} = H^{\mathrm{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)\,,$$

$$i\partial_t \psi_t^{\mathrm{BF}} = H^{\mathrm{BF}} \psi_t^{\mathrm{BF}}\,,$$

on $L^2(\mathbb{R}_x^3) \otimes \mathcal{F}(\{\varphi_t\}^\perp)$.

$H^{\mathrm{Bog}} \sim d\Gamma(\xi)$ Bogoliubov Hamiltonian, modelling free excitations. Q_t projects into $\{\varphi_t\}^\perp$. $a^{\#}(Q_t W_x \varphi_t)$ **creates or annihilates excitation** due to interaction of impurity with condensate.



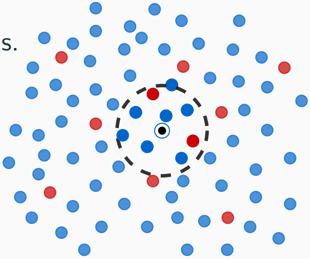
Bose Polaron

Bose Polaron: Quasi-particle of impurity and bosons.
It is described by effective dynamics generated by **Bogoliubov-Fröhlich** Hamiltonian H^{BF}

$$H^{\mathrm{BF}} = H^{\mathrm{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)\,,$$

$$i\partial_t \psi_t^{\mathrm{BF}} = H^{\mathrm{BF}} \psi_t^{\mathrm{BF}}\,,$$

on $L^2(\mathbb{R}^3_x) \otimes \mathcal{F}(\{\varphi_t\}^\perp)$.



Importance of the **Bogoliubov-Fröhlich** Hamiltonian H^{BF}:

- Its validity proves the **formation** of the **Bose Polaron**.
- We started with Schrödinger's equation and now consider a QFT of **matter** (impurity) interacting with a **field of excitations**.
- It **simplifies** the **dynamics**: "Free" quantum field interacting with matter.

Next Step

1. Motivation: Impurity Particles in Experiments
2. System Set-Up: Bose Gas in Condensation
3. Methods: Impurity Localization
4. Main Theorem: Validity Bose Polaron Dynamics

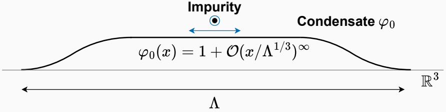
Methods Overview

$$\underbrace{H^{\text{ex}}}_{\text{Full excitation dynamics}} = \underbrace{H^{\text{BF}}}_{\text{Polaron dynamics}} + 1/\sqrt{\rho} \cdot \text{error} + \underbrace{\sqrt{\rho} W \star |\varphi_t|^2(x)}_{\text{Mean-field impurity-condensate interaction}} .$$

We want to show

$$H^{\text{ex}} \xrightarrow{\Lambda,\rho\to\infty} H^{\text{BF}} .$$

- Control **excitation number** $\to \left(1/\sqrt{\rho} \cdot \text{error}\right)$ small.
- Control **impurity localization** and φ_t remains **flat around the origin** $\to \sqrt{\rho} W \star |\varphi_t|^2(x) \sim \sqrt{\rho} W \star 1$.



Initial conditions

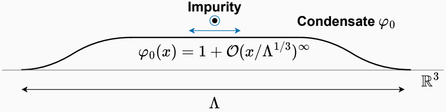
Methods Overview

$$\underbrace{H^{\text{ex}}}_{\text{Full excitation dynamics}} = \underbrace{H^{\text{BF}}}_{\text{Polaron dynamics}} + 1/\sqrt{\rho} \cdot \text{error} + \underbrace{\sqrt{\rho} W \star |\varphi_t|^2(x)}_{\text{Mean-field impurity-condensate interaction}} .$$

We want to show

$$H^{\text{ex}} \xrightarrow{\Lambda, \rho \to \infty} H^{\text{BF}} .$$

- Control **excitation number** $\to \left(1/\sqrt{\rho} \cdot \text{error}\right)$ small.
- Control **impurity localization** and φ_t remains **flat around the origin** $\to \sqrt{\rho} W \star |\varphi_t|^2(x) \sim \sqrt{\rho} W \star 1$.

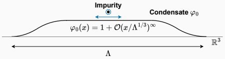


Initial conditions

Impurity Localization

Goal: Prove **impurity remains** in the **bulk** of the condensate:
$\forall T \geq 0,\ M \in \mathbb{N}_0\ \exists C_M > 0$ such that for all densities $\rho \geq 1$, volumes $\Lambda \geq 1$

$$\sup_{t\in[-T,T]} \| \underbrace{|x|^{2M}}_{\text{Impurity position}} \cdot \underbrace{\psi_t^{\mathrm{BF}}}_{\text{Polaron dynamics}} \| \leq C_M\,.$$



Initial conditions

Control **impurity position** x

$\downarrow$ needs

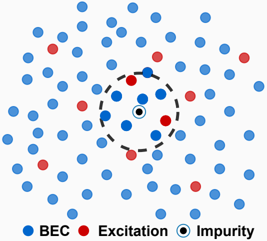
Control **kinetic energy** ∇_x

$\downarrow$

Control energy gained by excitation

$\downarrow$

Control **number of excitations** effectively interacting with impurity



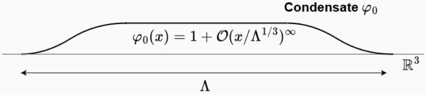
Next Step

1. Motivation: Impurity Particles in Experiments
2. System Set-Up: Bose Gas in Condensation
3. Methods: Impurity Localization
4. Main Theorem: Validity Bose Polaron Dynamics

Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:



Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, η *independent of* ρ, Λ *and* φ_0 **flat around the origin**.

Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, η *independent of* ρ, Λ *and* φ_0 **flat around the origin**.

ii) **Excitation number**$\leq$**Volume**.

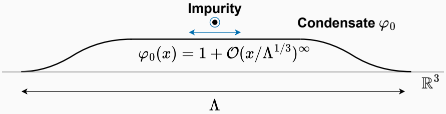
Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, *$\eta$ independent of ρ, Λ and φ_0* **flat around the origin**.

ii) **Excitation number**$\leq$**Volume**.



Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}}y)$, η *independent of* ρ, Λ *and* φ_0 **flat around the origin***:*

ii) **Excitation number**$\leq$**Volume***:*

iii) **Impurity-localized** *around origin:*

Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, η *independent of* ρ, Λ *and* φ_0 **flat around the origin***:* $\varphi_0(0) = 1$, $D^\beta \varphi_0(0) = 0$ *for all* $|\beta| \geq 1$.

ii) **Excitation number$\leq$Volume***:*

iii) **Impurity-localized** *around origin:*

Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, *$\eta$ independent of ρ, Λ and φ_0* **flat around the origin***: $\varphi_0(0) = 1$, $D^\beta \varphi_0(0) = 0$ for all $|\beta| \geq 1$.*

ii) **Excitation number≤Volume***: $\left\langle \psi_0^{\mathrm{BF}}, (\mathcal{N}_+ + 1)^n \psi_0^{\mathrm{BF}} \right\rangle \leq C_n \Lambda^n$, $0 \leq n \leq 4$.*

iii) **Impurity-localized** *around origin:*

Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, η *independent of* ρ, Λ *and* φ_0 **flat around the origin***:* $\varphi_0(0) = 1$, $D^\beta \varphi_0(0) = 0$ *for all* $|\beta| \geq 1$.

ii) **Excitation number≤Volume***:* $\left\langle \psi_0^{\mathrm{BF}}, (\mathcal{N}_+ + 1)^n \psi_0^{\mathrm{BF}} \right\rangle \leq C_n \Lambda^n$, $0 \leq n \leq 4$.

iii) **Impurity-localized** *around origin:*
$\left\langle \psi_0^{\mathrm{BF}}, (-\Delta_x + x^2 + U_{\mathcal{V}}(\mathcal{N}_+ + 1) U_{\mathcal{V}}^*)^M \psi_0^{\mathrm{BF}} \right\rangle \leq C_M$, *for all* $M \geq 0$.

Main Theorem: Validity Bose Polaron Dynamics

Theorem (Validity Bose Polaron Dynamics)

If we have the initial conditions:

i) **Condensate** *is a* **rescaled function** $\varphi_0(y) = \eta(\Lambda^{-\frac{1}{3}} y)$, η *independent of* ρ, Λ *and* φ_0 **flat around the origin**: $\varphi_0(0) = 1$, $D^\beta \varphi_0(0) = 0$ *for all* $|\beta| \geq 1$.

ii) **Excitation number≤Volume**: $\left\langle \psi_0^{\mathrm{BF}}, (\mathcal{N}_+ + 1)^n \psi_0^{\mathrm{BF}} \right\rangle \leq C_n \Lambda^n$, $0 \leq n \leq 4$.

iii) **Impurity-localized** *around origin:*
$\left\langle \psi_0^{\mathrm{BF}}, (-\Delta_x + x^2 + U_{\mathcal{V}}(\mathcal{N}_+ + 1) U_{\mathcal{V}}^*)^M \psi_0^{\mathrm{BF}} \right\rangle \leq C_M$, *for all* $M \geq 0$.

Then for $\psi_0^{\mathrm{ex}} = \psi_0^{\mathrm{BF}}$ *and for all times* $T \geq 0$ *there exists a constant* $C > 0$ *such that for all densities* $\rho \geq 1$, *volumes* $\Lambda \geq 1$

$$\sup_{t \in [-T, T]} \| \underbrace{e^{it(\rho^{1/2} \int W - \mu_0)} \psi_t^{\mathrm{ex}}}_{\text{Full excitation dynamics}} - \underbrace{\psi_t^{\mathrm{BF}}}_{\text{Polaron dynamics}} \|_{\mathcal{F}} \leq C \left(\frac{\Lambda^3}{\rho} \right)^{1/2} .$$

Convergence to 0 for $\rho, \Lambda \to \infty$, *with* $\Lambda^3 \ll \rho$. *We conclude* **effective description** *of the* **full microscopic dynamics** *through* $\mathbf{H}^{\mathrm{BF}}$.

Conclusion

- **Bose Polaron**: Quasi-particle of impurity and bosons.
- We proved the **effective description** of the **full microscopic dynamics** through the **Polaron dynamics**:
  $H^{\mathrm{BF}} = H^{\mathrm{Bog}} - \frac{\Delta_x}{2m} + a(Q_t W_x \varphi_t) + a^*(Q_t W_x \varphi_t)$.
- **Impurity** is **localized** in the large Bose gas.

- (Ongoing) Derive effective limiting **dynamics** independent of Λ, ρ at **infinite volume**.

Thank you for your attention!

Literature

Derivation of the Bogoliubov-Fröhlich Hamiltonian from the microscopic dynamics:

Myśliwy-Seiringer 2020: **Spectrum** at low energies on the unit torus in the **mean-field scaling** (Interactions are often but weak).

Lampart-Pickl 2022: **Dynamics** on the unit torus in the **mean-field scaling**.

Lampart-Triay 2024: **Spectrum** on the unit torus in the **dilute scaling** (Interactions are rare but strong).

Literature

Derivation of the Bogoliubov-Fröhlich Hamiltonian from the microscopic dynamics:

Myśliwy-Seiringer 2020: Spectrum at low energies on the unit torus in the **mean-field scaling** (Interactions are often but weak).

Lampart-Pickl 2022: Dynamics on the unit torus in the **mean-field scaling**.

Lampart-Triay 2024: Spectrum on the unit torus in the **dilute scaling** (Interactions are rare but strong).

Our work extents the results of **[Lampart-Pickl 2022]** to **large volumes** and without periodic boundary condition leading to a **non-constant condensate**.

Generalized Initial State

Goal: Initial state with

- $\mathcal{O}(1)$ excitations locally (in unit volume)
- $\mathcal{O}(\Lambda)$ excitations globally (in volume Λ)

Currently: ψ_0^{BF} with $\mathcal{O}(1)$ excitations globally $\to$ $\mathcal{O}(1)$ excitations locally.
Needed for tracer localization! ($\|(-\Delta_x + x^2 + \mathcal{N} + 1)^M \psi_0^{\mathrm{BF}}\| \leq C$)
Solution: Bogoliubov transformed initial state

$$U_{\mathcal{V}} \psi_0^{\mathrm{BF}}$$

with

$$\underbrace{\|\mathcal{V}\|_{\mathrm{op}} \leq C}_{\mathcal{O}(1) \text{ local excitations}} \quad , \quad \underbrace{\|\mathcal{V}\mathcal{V}^* - 1\|_{HS} \leq C \cdot \Lambda}_{\mathcal{O}(\Lambda) \text{ global excitations}} .$$

Same estimates as before!

$$\|(-\Delta_x + x^2 + U_{\mathcal{V}}(\mathcal{N}+1)U_{\mathcal{V}}^*)^M \psi_t^{\mathrm{BF}}\| \leq C \text{ if } \psi_{t=0}^{\mathrm{BF}} = U_{\mathcal{V}}\psi_0^{\mathrm{BF}},$$
$$\|(-\Delta_x + x^2 + U_{\mathcal{V}}(\mathcal{N}+1)U_{\mathcal{V}}^*)^M U_{\mathcal{V}}\psi_0^{\mathrm{BF}}\| \leq C.$$

Scaling

Density $\rho \geq 1$ and $\Lambda = \rho^{\alpha}$, the gas is for all $\alpha > 0$ dense!

- The case $\alpha = 0$ is the standard mean-field scaling with fixed volume (e.g., $\Lambda = 1$).
- The case $\alpha = \infty$ corresponds to the thermodynamic limit with constant density (e.g. $\rho = 1$) as the volume grows.
- Our approximation is valid for $0 < \alpha < 1/3$.

Comparison with the β-scaling:

$$-\sum_{i} \frac{\Delta_{y_i}}{N^{2\beta}} + N^{3\beta-1} \sum_{i,j} V\left(N^{\beta}(y_i - y_j)\right), \quad y_i \in [-1/2, 1/2].$$

- $\beta := \frac{\alpha}{3(1+\alpha)}$, $0 < \beta < 1/4$.